1 Review Problems

Problem 1.1. Let X and Y have joint probability function as follows:

$f(x,y)$	$y=0$	$y=2$	$y=4$
$x=0$	0.1	0.2	c
$x=1$	0.2	k	0.1
$x=2$	0.2	0	0.1

Suppose we also know that $\mathbb{E}(Y) = 1.6$ What is the value of k?

Solution 1.1. There are two unknowns in this problem k and c.

The joint probability sums to 1, so summing all the values in the table, we get

$$0.1 + 0.2 + 0.2 + 0.2 + k + c + 0.1 + 0.1 = 1 \Rightarrow c = 0.1 - k$$

Next, we know that $\mathbb{E}(Y) = 1.6$. Computing the marginals of Y gives us $\mathbb{P}(Y = 0) = 0.1 + 0.2 + 0.2 = 0.5$, $\mathbb{P}(Y = 2) = 0.2 + k + 0$ and $\mathbb{P}(Y = 4) = 0.1 + 0.1 + c$. Hence, the expected value of Y is

$$1.6 = \mathbb{E}(Y) = 0.5 \cdot 0 + (0.2 + k) \cdot 2 + (0.2 + c) \cdot 4$$

Plugging in $c = 0.1 - k$ gives

$$1.6 = E(Y) = 0 + (0.2 + k) \cdot 2 + (0.2 + 0.1 - k) \cdot 4 \Rightarrow k = 0.$$

Problem 1.2. Suppose the random variable X takes the values 0, 1 and 2 with probabilities 0.3, 0.2 and 0.5, respectively. What is the moment generating function of X?

Solution 1.2. Since X is discrete and finitely supported, by the definition of the MGF,

$$M_X(t) = \mathbb{E}(e^{tX}) = \sum_{\text{all } x} e^{tx}\, \mathbb{P}(X = x) = e^0 \cdot 0.3 + e^t \cdot 0.2 + e^{2t} \cdot 0.5.$$

Remark 1. When X has finite support, we can also easily read off what the PMF is from the MGF. The coefficients in front of the exponential tell us the probabilities, and the coefficient in front of the t gives the corresponding value of x,

$$M_X(t) = e^{0} \cdot 0.3 + e^{1 \cdot t} \cdot 0.2 + e^{2 \cdot t} \cdot 0.5$$

implies that $\mathbb{P}(X = 0) = 0.3$, $\mathbb{P}(X = 1) = 0.2$, $\mathbb{P}(X = 2) = 0.5$.

Problem 1.3. Suppose X follows the continuous uniform distribution on the interval $(2, b)$ (where $b > 2$ is to be determined) and you know that $\mathbb{P}(X > 6) = 0.6$. What is the value of b?

Solution 1.3. Since $X \sim \mathrm{U}(2, b)$ we know that

$$F_X(x) = \mathbb{P}(X \leq x) = \frac{1}{b-2} \int_2^x dt = \frac{x-2}{b-2}.$$

We are given that $\mathbb{P}(X > 6) = 1 - F_X(6) = 0.6$. Solving gives

$$F_X(6) = 0.4 = \frac{6-2}{b-2} \Leftrightarrow b = 12$$

Problem 1.4. Denote by $\Phi(x) = \mathbb{P}(Z \le x)$ for $Z \sim N(0,1)$ the cumulative distribution function of the standard normal distribution. Let $Y = 1/|Z|$. What is the CDF of Y?

Solution 1.4. The random variable Y can only take non-negative values. Using symmetry of the standard normal, we get

$$\begin{aligned}\mathbb{P}(1/|Z| \le x) &= \mathbb{P}(1/x \le |Z|)\\ &= P(Z \ge 1/x) + P(Z \le -1/x)\\ &= 1 - \Phi(1/x) + \Phi(-1/x)\\ &= 1 - \Phi(1/x) + 1 - \Phi(1/x)\\ &= 2(1 - \Phi(1/x)). \quad x > 0\end{aligned}$$

Problem 1.5. A committee of 4 people is to be formed by randomly selecting from a group of 6 men and 3 women. Suppose that we are concerned that there may be a gender imbalance on the committee. Let M and W represent the number of men and women selected to be on the committee, respectively. Let $D = M - W$. What is $\mathbb{E}(D)$?

Solution 1.5. Let A_i be the event that the ith person is a man. We have

$$\mathbb{P}(A_i) = \frac{6}{9} = \frac{2}{3}$$

for $i = 1, 2, 3, 4$. Therefore, $M = \mathbb{1}_{A_1} + \mathbb{1}_{A_2} + \mathbb{1}_{A_3} + \mathbb{1}_{A_4}$ has expected value

$$\mathbb{E}(M) = E\left(\sum_{i=1}^{4} \mathbb{1}_{A_i}\right) = \sum_{i=1}^{4} \mathbb{E}(\mathbb{1}_{A_i}) = \sum_{i=1}^{4} \mathbb{P}(A_i) = 4 \cdot \frac{2}{3} = \frac{8}{3}$$

We know the committee has 4 members, thus $W = 4 - M$. Hence,

$$\mathbb{E}(D) = \mathbb{E}(M - W) = \mathbb{E}(M - (4 - M)) = \mathbb{E}(2M - 4) = 2\,\mathbb{E}(M) - 4 = 2 \cdot \frac{8}{3} - 4 = \frac{4}{3}.$$

Problem 1.6. Suppose that a city contains 330,000 adults. Their heights are normally distributed with a mean of 175 cm and a variance of 100 cm^2. How tall do you need to be at least in order to be taller than 250,000 of the adults in this city?

Solution 1.6. Let $X \sim N(175, 10^2)$ denote the random height of an adult. We want the $\frac{250000}{330000} = 0.76$ quantile of the distribution of X. We have

$$F_Z^{-1}(0.76) = 0.7063.$$

Therefore, using the standardization trick, we have $F_X^{-1}(p) = \mu + \sigma F_Z^{-1}(p) = 175 + 10 \cdot 0.7063 = 182.063$.

Problem 1.7. Let X and Y be random variables with $\mathrm{Var}(X) = \sigma_X^2$, $\mathrm{Var}(Y) = \sigma_Y^2$ and $\mathrm{Cov}(X, Y) = \rho\sigma_X\sigma_Y$ for some $\rho \in (-1, 1)$.

1. What is $\mathrm{Cov}(X - Y, X - Y)$?
2. What is $\mathrm{Cov}(X - Y, X + Y)$?

Solution 1.7. Both problems are solved by an application of the binlinearity formula,

$$\begin{aligned}\mathrm{Cov}(X-Y,X-Y) &= \mathrm{Cov}(X,X) - \mathrm{Cov}(X,Y) - \mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,Y)\\ &= \mathrm{Var}(X) - 2\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)\\ &= \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(X-Y,X+Y) &= \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(Y,X) - \mathrm{Cov}(Y,Y)\\ &= \mathrm{Var}(X) - \mathrm{Var}(Y)\\ &= \sigma_X^2 - \sigma_Y^2.\end{aligned}$$