1 Sampling from the Normal Distribution

The normal distribution is the most commonly assumed population in statistics. In this case, we can compute the sampling distributions of $\bar{X}$ and S^2 exactly, and they lead to two new distributions that appear everywhere in statistical inference.

1.1 The Chi-Squared Distribution

Definition 1 (Chi-Squared Distribution)**.** A random variable V has a chi-squared distribution with p degrees of freedom, denoted $V \sim \chi^2_p$, if it has PDF

$$f_V(v) = \frac{1}{\Gamma(p/2)2^{p/2}} v^{\frac{p}{2}-1} e^{-\frac{v}{2}}, \qquad v > 0.$$

In other words, χ^2_p is the $\Gamma(\frac{p}{2}, 2)$ distribution, so its MGF is $M_V(t) = (1-2t)^{-\frac{p}{2}}$ for $t < \frac{1}{2}$.

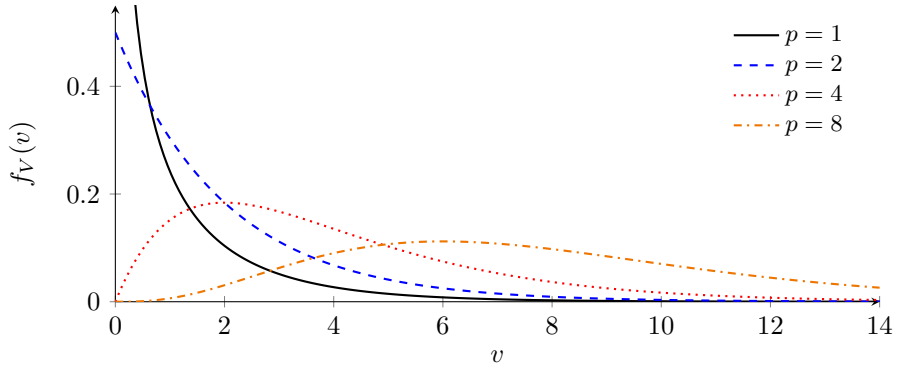


Figure: The PDF of the χ^2_p distribution for several degrees of freedom p.

The chi-squared distribution is important because it is the distribution of the sum of squares of independent standard normals.

Lemma 1 (*Properties of the Chi-Squared Distribution*)

1. *If* $Z \sim \mathrm{N}(0,1)$*, then* $Z^2 \sim \chi^2_1$*.*
2. *If* $V_1, \ldots, V_k$ *are independent and* $V_i \sim \chi^2_{p_i}$*, then* $V_1 + \cdots + V_k \sim \chi^2_{p_1+\cdots+p_k}$*.*
3. *If* $V \sim \chi^2_p$*, then* $\mathbb{E}[V] = p$ *and* $\mathrm{Var}(V) = 2p$*.*

Corollary 1

If $Z_1, \ldots, Z_p$ *are i.i.d.* $\mathrm{N}(0,1)$*, then* $Z_1^2 + \cdots + Z_p^2 \sim \chi^2_p$*.*

Remark 1. Geometrically, χ^2_p is the distribution of the squared length $\|Z\|_2^2$ of a standard normal vector $Z \sim \mathrm{N}(0, I) \in \mathbb{R}^p$. The degrees of freedom are the dimension of the space the normal vector lives in.

1.1.1 The Sample Mean and Variance

The following is the fundamental theorem about normal random samples.

Theorem 1 (*Sampling from the Normal Distribution*)

Let $X_1, \dots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$. Then

1. *$\bar{X}$ and S^2 are independent,*
2. *$\bar{X} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n})$,*
3. *$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.*

Remark 2. The independence is surprising since $\bar{X}$ appears in the formula for S^2. It is a very special property of the normal distribution. In fact, a theorem of Lukacs shows that if a population with finite variance has $\bar{X}$ and S^2 independent, then the population must be normal. The proof (Problem 1.8) uses the rotational invariance of the standard normal vector from Week 5, and it explains the $n-1$ degrees of freedom: $\bar{X}$ captures the component of the data in the direction $(1, \dots, 1)$, and S^2 captures the length of the data in the remaining $n-1$ orthogonal directions.

Remark 3. Notice that the distribution of $\frac{(n-1)S^2}{\sigma^2}$ does not depend on the unknown mean μ, so we can use it to make probability statements about S^2 even if μ is unknown (Problem 1.1).

1.2 Student's t Distribution

If $X_1, \dots, X_n$ is a random sample from $\mathrm{N}(\mu, \sigma^2)$, then Theorem 1 implies that

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1).$$

This is useful for making statements about μ only if we know σ. In practice, σ is also unknown, so it is natural to replace it with the sample standard deviation S. To find the distribution of the resulting quantity, we rewrite it as

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{(\bar{X} - \mu)/(\sigma/\sqrt{n})}{\sqrt{S^2/\sigma^2}} = \frac{Z}{\sqrt{V/(n-1)}}$$

where $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$ and $V = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ are independent by Theorem 1. This motivates the following definition.

Definition 2 (Student's t Distribution)**.** Let $Z \sim \mathrm{N}(0,1)$ and $V \sim \chi^2_p$ be independent. The random variable

$$T = \frac{Z}{\sqrt{V/p}}$$

has a Student's t distribution with p degrees of freedom, denoted $T \sim t_p$.

Theorem 2 (*PDF of the t Distribution*)

If $T \sim t_p$, then T has PDF

$$f_T(t) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{\sqrt{p\pi}} \left(1 + \frac{t^2}{p}\right)^{-\frac{p+1}{2}}, \qquad t \in \mathbb{R}.$$

Corollary 2

If $X_1, \ldots, X_n$ is a random sample from $\mathrm{N}(\mu, \sigma^2)$, then

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

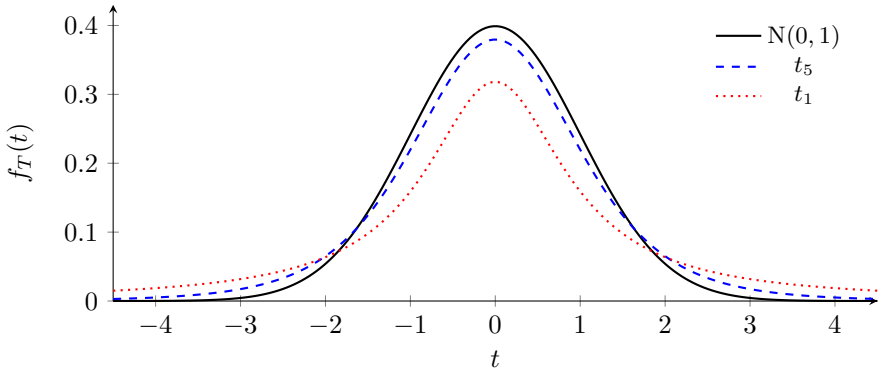


Figure: The PDF of the t_1, t_5, and standard normal distributions. The t distributions have heavier tails.

Remark 4. The t distribution looks like the standard normal, but it has heavier tails since replacing σ by S adds extra randomness. Some important properties are:

1. The t_1 distribution is the Cauchy distribution, so it has no mean.
2. $T \sim t_p$ only has finite moments of order less than p. We have $\mathbb{E}[T] = 0$ if $p > 1$ and $\mathrm{Var}(T) = \frac{p}{p-2}$ if $p > 2$ (Problem 1.4).
3. As $p \to \infty$, $t_p \to \mathrm{N}(0, 1)$ in distribution. This is consistent with the strong law of large numbers since $S^2 = \frac{n}{n-1}(\frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2) \to \sigma^2$ almost surely, so replacing σ by S makes little difference for large n. One can also check directly that the PDF converges pointwise to the standard normal PDF using Stirling's formula.

1.3 Snedecor's F Distribution

Suppose we have samples from two normal populations and we want to compare their variability. The natural statistic is the ratio of their sample variances, which leads to the following distribution.

Definition 3 (Snedecor's F Distribution)**.** Let $U \sim \chi_p^2$ and $V \sim \chi_q^2$ be independent. The random variable

$$F = \frac{U/p}{V/q}$$

has a Snedecor's F distribution with p and q degrees of freedom, denoted $F \sim F_{p,q}$.

A computation similar to the proof of Theorem 2 shows that $F \sim F_{p,q}$ has PDF

$$f_F(x) = \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\left(\frac{p}{q}\right)^{\frac{p}{2}} \frac{x^{\frac{p}{2}-1}}{(1 + \frac{p}{q}x)^{\frac{p+q}{2}}}, \qquad x > 0.$$

Corollary 3

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu_X, \sigma_X^2)$ and let $Y_1, \ldots, Y_m$ be an independent random sample from $\mathrm{N}(\mu_Y, \sigma_Y^2)$. If S_X^2 and S_Y^2 denote the sample variances, then

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F_{n-1,m-1}.$$

The F distribution is related to the other distributions we have seen.

Theorem 3 (*Properties of the F Distribution*)

1. *If $X \sim F_{p,q}$, then $\frac{1}{X} \sim F_{q,p}$.*
2. *If $T \sim t_q$, then $T^2 \sim F_{1,q}$.*
3. *If $X \sim F_{p,q}$, then $\frac{(p/q)X}{1+(p/q)X} \sim \mathrm{Beta}(\frac{p}{2}, \frac{q}{2})$.*

1.4 Example Problems

Problem 1.1. Let $X_1, \ldots, X_{10}$ be a random sample from $\mathrm{N}(\mu, 4)$ where μ is unknown.

1. Compute $\mathbb{P}(S^2 > 6)$.
2. Find the constant c such that $\mathbb{P}(S^2 \leq c) = 0.95$.

Solution 1.1. By Theorem 1, $V = \frac{9S^2}{4} \sim \chi_9^2$, regardless of the value of μ.

Part 1: Using a χ^2 table (or software),

$$\mathbb{P}(S^2 > 6) = \mathbb{P}\left(\frac{9S^2}{4} > \frac{9 \cdot 6}{4}\right) = \mathbb{P}(V > 13.5) = 0.141.$$

Part 2: We have

$$0.95 = \mathbb{P}(S^2 \leq c) = \mathbb{P}\left(V \leq \frac{9c}{4}\right).$$

The 0.95 quantile of the χ_9^2 distribution is 16.919, so $\frac{9c}{4} = 16.919$, which gives $c = 7.52$.

Problem 1.2. Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$. Compute $\mathrm{Var}(S^2)$.

Solution 1.2. By Theorem 1, $V = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$, so $\mathrm{Var}(V) = 2(n-1)$ by Lemma 1. Since $S^2 = \frac{\sigma^2}{n-1}V$,

$$\mathrm{Var}(S^2) = \frac{\sigma^4}{(n-1)^2}\mathrm{Var}(V) = \frac{\sigma^4}{(n-1)^2}2(n-1) = \frac{2\sigma^4}{n-1}.$$

Remark 5. Combined with Chebyshev's inequality (Week 7), this shows that S^2 concentrates around σ^2,

$$\mathbb{P}(|S^2 - \sigma^2| \geq \varepsilon) \leq \frac{\mathrm{Var}(S^2)}{\varepsilon^2} = \frac{2\sigma^4}{(n-1)\varepsilon^2} \to 0.$$

Problem 1.3. Let $X_1, \ldots, X_{16}$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$, where μ and σ^2 are both unknown.

1. Compute the probability that $\bar{X}$ is within $\frac{S}{2}$ of μ.
2. Compare this with the probability that $\bar{X}$ is within $\frac{\sigma}{2}$ of μ.

Solution 1.3.

Part 1: By Corollary 2, $T = \frac{\bar{X}-\mu}{S/\sqrt{16}} = \frac{\bar{X}-\mu}{S/4} \sim t_{15}$. Therefore,

$$\mathbb{P}\left(|\bar{X}-\mu| \le \frac{S}{2}\right) = \mathbb{P}\left(\frac{|\bar{X}-\mu|}{S/4} \le 2\right) = \mathbb{P}(|T| \le 2) = 0.936$$

using a t table (or software).

Part 2: By Theorem 1, $Z = \frac{\bar{X}-\mu}{\sigma/4} \sim \mathrm{N}(0,1)$. Therefore,

$$\mathbb{P}\left(|\bar{X}-\mu| \le \frac{\sigma}{2}\right) = \mathbb{P}(|Z| \le 2) = 2\Phi(2) - 1 = 0.954.$$

The probability is smaller when we use S because the t_{15} distribution has heavier tails than the standard normal. For example, $\mathbb{P}(|T| \le x) = 0.95$ requires $x = 2.131$ instead of 1.96.

Problem 1.4. Let $T \sim t_p$. Show that $\mathbb{E}[T] = 0$ if $p > 1$ and $\mathrm{Var}(T) = \frac{p}{p-2}$ if $p > 2$.

Solution 1.4. Write $T = \sqrt{p}\, Z V^{-1/2}$ where $Z \sim \mathrm{N}(0,1)$ and $V \sim \chi^2_p$ are independent. For $k > 0$, independence implies $\mathbb{E}[|T|^k] = p^{k/2}\, \mathbb{E}[|Z|^k]\, \mathbb{E}[V^{-k/2}]$. All the moments of Z are finite, and

$$\mathbb{E}[V^{-k/2}] = \frac{1}{\Gamma(p/2)2^{p/2}} \int_0^\infty v^{\frac{p-k}{2}-1} e^{-\frac{v}{2}}\, dv.$$

The integral is finite if and only if $\frac{p-k}{2} > 0$ (the integrand is not integrable near 0 otherwise), i.e. $k < p$. In this case, the integral is the normalizing constant of the χ^2_{p-k} PDF, so

$$\mathbb{E}[V^{-k/2}] = \frac{\Gamma(\frac{p-k}{2})2^{\frac{p-k}{2}}}{\Gamma(\frac{p}{2})2^{\frac{p}{2}}} = \frac{\Gamma(\frac{p-k}{2})}{\Gamma(\frac{p}{2})2^{k/2}}.$$

Mean: If $p > 1$, then $\mathbb{E}[|T|] < \infty$ and $\mathbb{E}[T] = \sqrt{p}\, \mathbb{E}[Z]\, \mathbb{E}[V^{-1/2}] = 0$.

Variance: If $p > 2$, then using $\Gamma(\frac{p}{2}) = (\frac{p}{2}-1)\Gamma(\frac{p}{2}-1)$,

$$\mathrm{Var}(T) = \mathbb{E}[T^2] = p\, \mathbb{E}[Z^2]\, \mathbb{E}[V^{-1}] = p \cdot 1 \cdot \frac{\Gamma(\frac{p}{2}-1)}{2\Gamma(\frac{p}{2})} = \frac{p}{2(\frac{p}{2}-1)} = \frac{p}{p-2}.$$

Problem 1.5. Let $X_1, \ldots, X_{10}$ be a random sample from $\mathrm{N}(\mu_X, \sigma^2)$ and $Y_1, \ldots, Y_{10}$ be an independent random sample from $\mathrm{N}(\mu_Y, \sigma^2)$, so both populations have the same variance.

1. Compute $\mathbb{P}(S_X^2/S_Y^2 > 3)$.
2. Find a constant c such that $\mathbb{P}(\frac{1}{c} \le S_X^2/S_Y^2 \le c) = 0.90$.

Solution 1.5. Since $\sigma_X^2 = \sigma_Y^2$, the variances cancel in Corollary 3, so $F = \frac{S_X^2}{S_Y^2} \sim F_{9,9}$.

Part 1: Using an F table (or software), $\mathbb{P}(F > 3) = 0.0587$.

Part 2: Let c be the 0.95 quantile of $F_{9,9}$, which is $c = 3.179$, so $\mathbb{P}(F > c) = 0.05$. By Theorem 3, $\frac{1}{F} \sim F_{9,9}$ as well, so

$$\mathbb{P}\left(F < \frac{1}{c}\right) = \mathbb{P}\left(\frac{1}{F} > c\right) = 0.05.$$

Therefore, $\mathbb{P}(\frac{1}{3.179} \le F \le 3.179) = 1 - 0.05 - 0.05 = 0.90$.

Remark 6. Even when the two populations have exactly the same variance, one sample variance is more than 3 times the other with probability about 0.12 (by symmetry). With small samples, sample variances are quite noisy, which is consistent with Problem 1.2.

Problem 1.6. Let $U \sim \Gamma(a, \beta)$ and $V \sim \Gamma(b, \beta)$ be independent.

1. Show that $S = U + V$ and $R = \frac{U}{U+V}$ are independent with
$$S \sim \Gamma(a+b, \beta) \qquad \text{and} \qquad R \sim \text{Beta}(a, b).$$
2. Use Part 1 to give another proof of the third part of Theorem 3.

Solution 1.6. **Part 1:** By independence, the joint PDF of U and V is

$$f_{U,V}(u,v) = \frac{1}{\Gamma(a)\Gamma(b)\beta^{a+b}} u^{a-1} v^{b-1} e^{-(u+v)/\beta}, \qquad u, v > 0.$$

We use the same change of variables as in Assignment 1, Problem 1.6. The inverse transformation is $u = rs$ and $v = s(1-r)$ for $0 < r < 1$ and $s > 0$, with Jacobian

$$\left|\det \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{pmatrix}\right| = \left|\det \begin{pmatrix} s & r \\ -s & 1-r \end{pmatrix}\right| = s(1-r) + rs = s.$$

Therefore, the joint PDF of (R, S) is

$$\begin{aligned} f_{R,S}(r,s) &= \frac{1}{\Gamma(a)\Gamma(b)\beta^{a+b}} (rs)^{a-1} (s(1-r))^{b-1} e^{-s/\beta} \cdot s \\ &= \underbrace{\frac{1}{\Gamma(a+b)\beta^{a+b}} s^{a+b-1} e^{-s/\beta}}_{\Gamma(a+b,\beta)\text{ PDF}} \cdot \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} r^{a-1}(1-r)^{b-1}}_{\text{Beta}(a,b)\text{ PDF}} \end{aligned}$$

for $0 < r < 1$ and $s > 0$. The joint PDF factorizes, so by Week 5, Proposition 1, R and S are independent, and since the first factor is the $\Gamma(a+b, \beta)$ PDF, the second factor must be the PDF of R.

Part 2: Let $X = \frac{U/p}{V/q} \sim F_{p,q}$ where $U \sim \chi_p^2 = \Gamma(\frac{p}{2}, 2)$ and $V \sim \chi_q^2 = \Gamma(\frac{q}{2}, 2)$ are independent. Since $\frac{p}{q}X = \frac{U}{V}$,

$$\frac{(p/q)X}{1 + (p/q)X} = \frac{U/V}{1 + U/V} = \frac{U}{U+V} \sim \text{Beta}\left(\frac{p}{2}, \frac{q}{2}\right)$$

by Part 1. Notice that we did not need the PDF of the F distribution for this proof.

Remark 7. Since the second factor must integrate to 1, this argument also proves the identity $\int_0^1 r^{a-1}(1-r)^{b-1}\,dr = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ from Assignment 1. A consequence of Part 1 is that if $X_1, \ldots, X_n$ are i.i.d. Exp(θ), then the fraction $\frac{X_1}{X_1 + \cdots + X_n} \sim \text{Beta}(1, n-1)$ does not depend on θ, and is independent of the total $X_1 + \cdots + X_n$.

1.5 Proofs of Key Results

Problem 1.7. Prove Lemma 1 and Corollary 1.

Solution 1.7. We first check the MGF of $V \sim \chi^2_p$. For $t < \frac{1}{2}$,

$$M_V(t) = \frac{1}{\Gamma(p/2)2^{p/2}} \int_0^\infty v^{\frac{p}{2}-1} e^{-(\frac{1}{2}-t)v}\, dv = \frac{1}{\Gamma(p/2)2^{p/2}} \cdot \frac{\Gamma(p/2)}{(\frac{1}{2}-t)^{p/2}} = (1-2t)^{-\frac{p}{2}},$$

where we used the substitution $x = (\frac{1}{2}-t)v$ and the definition of the Gamma function.

Part 1: For $t < \frac{1}{2}$,

$$M_{Z^2}(t) = \int_{-\infty}^\infty e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-2t)z^2}{2}}\, dz = \frac{1}{\sqrt{1-2t}} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\, dw = (1-2t)^{-\frac{1}{2}}$$

using the substitution $w = \sqrt{1-2t}z$. This is the MGF of χ^2_1, so $Z^2 \sim \chi^2_1$ by the uniqueness of MGFs.

Part 2: By the independence property of MGFs,

$$M_{V_1+\cdots+V_k}(t) = \prod_{i=1}^k M_{V_i}(t) = \prod_{i=1}^k (1-2t)^{-\frac{p_i}{2}} = (1-2t)^{-\frac{p_1+\cdots+p_k}{2}},$$

which is the MGF of $\chi^2_{p_1+\cdots+p_k}$.

Part 3: Differentiating the MGF,

$$M_V'(t) = p(1-2t)^{-\frac{p}{2}-1} \qquad \text{and} \qquad M_V''(t) = p(p+2)(1-2t)^{-\frac{p}{2}-2},$$

so $\mathbb{E}[V] = M_V'(0) = p$ and $\mathrm{Var}(V) = M_V''(0) - M_V'(0)^2 = p(p+2) - p^2 = 2p$.

Corollary: By Part 1, $Z_1^2, \ldots, Z_p^2$ are independent χ^2_1 random variables, so by Part 2, their sum is χ^2_p.

Problem 1.8. Prove Theorem 1.

Solution 1.8.

Step 1: As usual, we first standardize to reduce the problems to one involving standard normals. Let $Z_i = \frac{X_i - \mu}{\sigma}$, so $Z_1, \ldots, Z_n$ are i.i.d. $\mathrm{N}(0,1)$ and $Z = (Z_1, \ldots, Z_n) \sim \mathrm{N}(0, I)$. Since $X_i = \mu + \sigma Z_i$, we have

$$\bar{X} = \mu + \sigma \bar{Z} \qquad \text{and} \qquad X_i - \bar{X} = \sigma(Z_i - \bar{Z}) \implies \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^n (Z_i - \bar{Z})^2.$$

Therefore, it suffices to show that $\bar{Z} \sim \mathrm{N}(0, \frac{1}{n})$ and $\sum_{i=1}^n (Z_i - \bar{Z})^2 \sim \chi^2_{n-1}$ are independent.

Step 2: By rotation invariance, we can pick a clever rotation to recover the empirical average. Let Q be an $n \times n$ orthogonal matrix ($QQ^\intercal = Q^\intercal Q = I$) whose first row is $q_1 = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)$. For example, we can take the so called Helmert matrix whose rows for $k = 2, \ldots, n$ are

$$q_k = \frac{1}{\sqrt{k(k-1)}} (\underbrace{1, \ldots, 1}_{k-1}, -(k-1), 0, \ldots, 0).$$

Let $W = QZ$. Since $Z = Q^\intercal W$ and $|\det(Q^\intercal)| = 1$, the change of variables formula implies that

$$f_W(w) = f_Z(Q^\intercal w) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\|Q^\intercal w\|_2^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\|w\|_2^2},$$

since orthogonal matrices preserve lengths. Therefore $W \sim \mathrm{N}(0, I)$, so $W_1, \dots, W_n$ are i.i.d. $\mathrm{N}(0,1)$ (Week 5, Remark 17). This is the rotational invariance of the standard normal from Week 5, Remark 13.

Step 3: We now compute the sum of squares. The first coordinate is $W_1 = q_1 \cdot Z = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i = \sqrt{n}\bar{Z}$. Since $\|W\|_2^2 = \|Z\|_2^2$, the sum identities

$$\sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2$$

implies that

$$\sum_{k=2}^n W_k^2 = \sum_{i=1}^n Z_i^2 - W_1^2 = \sum_{i=1}^n Z_i^2 - n\bar{Z}^2 = \sum_{i=1}^n (Z_i - \bar{Z})^2.$$

Since W_1 is independent of $(W_2, \dots, W_n)$, $\bar{Z} = \frac{W_1}{\sqrt{n}}$ is independent of $\sum_{i=1}^n (Z_i - \bar{Z})^2 = \sum_{k=2}^n W_k^2$. Furthermore, $\bar{Z} = \frac{W_1}{\sqrt{n}} \sim \mathrm{N}(0, \frac{1}{n})$, and $\sum_{k=2}^n W_k^2 \sim \chi^2_{n-1}$ by Corollary 1 since it is the sum of $n-1$ squared i.i.d. standard normals.

Remark 8. Geometrically, $\sqrt{n}\bar{Z}$ is the length of the projection of Z onto the direction $(1, \dots, 1)$, and $\sum_i (Z_i - \bar{Z})^2$ is the squared length of the projection onto the $(n-1)$-dimensional orthogonal complement (compare with the projection picture in Week 6, Problem 1.2). For a standard normal vector, the projections onto orthogonal subspaces are independent, and the squared length of the projection onto a k-dimensional subspace is χ^2_k.

Problem 1.9. Prove Theorem 2.

Solution 1.9. Let $Z \sim \mathrm{N}(0,1)$ and $V \sim \chi^2_p$ be independent, so their joint PDF is

$$f_{Z,V}(z,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{1}{\Gamma(p/2) 2^{p/2}} v^{\frac{p}{2}-1} e^{-\frac{v}{2}}, \qquad z \in \mathbb{R}, v > 0.$$

Consider the transformation $(T, V) = (\frac{Z}{\sqrt{V/p}}, V)$, whose inverse is $z = t\sqrt{v/p}$ and $v = v$. The Jacobian of the inverse is

$$\left|\det \begin{pmatrix} \frac{\partial z}{\partial t} & \frac{\partial z}{\partial v} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial v} \end{pmatrix}\right| = \left|\det \begin{pmatrix} \sqrt{v/p} & \frac{t}{2\sqrt{vp}} \\ 0 & 1 \end{pmatrix}\right| = \sqrt{\frac{v}{p}},$$

so the joint PDF of (T, V) is

$$f_{T,V}(t,v) = f_{Z,V}\big(t\sqrt{v/p}, v\big)\sqrt{\frac{v}{p}} = \frac{1}{\sqrt{2\pi p}\,\Gamma(p/2) 2^{p/2}} v^{\frac{p+1}{2}-1} e^{-\frac{v}{2}(1+\frac{t^2}{p})}.$$

To get the marginal PDF of T, we integrate out v. The integrand is a $\Gamma(\frac{p+1}{2}, \frac{2}{1+t^2/p})$ PDF without its normalizing constant, so

$$\int_0^\infty v^{\frac{p+1}{2}-1} e^{-\frac{v}{2}(1+\frac{t^2}{p})}\, dv = \Gamma\Big(\frac{p+1}{2}\Big)\Big(\frac{2}{1+t^2/p}\Big)^{\frac{p+1}{2}}.$$

Therefore,

$$f_T(t) = \frac{\Gamma(\frac{p+1}{2}) 2^{\frac{p+1}{2}}}{\sqrt{2\pi p}\,\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} \left(1 + \frac{t^2}{p}\right)^{-\frac{p+1}{2}} = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{\sqrt{p\pi}} \left(1 + \frac{t^2}{p}\right)^{-\frac{p+1}{2}}.$$

Remark 9. When $p = 1$, we have $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, so $f_T(t) = \frac{1}{\pi(1+t^2)}$, which is the Cauchy PDF.

Problem 1.10. Prove Theorem 3.

Solution 1.10.

Part 1: If $X = \frac{U/p}{V/q}$ with $U \sim \chi^2_p$ and $V \sim \chi^2_q$ independent, then $\frac{1}{X} = \frac{V/q}{U/p}$, which is $F_{q,p}$ by definition.

Part 2: If $T = \frac{Z}{\sqrt{V/q}}$ with $Z \sim \mathrm{N}(0,1)$ and $V \sim \chi^2_q$ independent, then

$$T^2 = \frac{Z^2/1}{V/q}.$$

Since $Z^2 \sim \chi^2_1$ (Lemma 1) is independent of V, we have $T^2 \sim F_{1,q}$ by definition.

Part 3: Let $Y = g(X) = \frac{(p/q)X}{1+(p/q)X}$. The function g is increasing from $(0,\infty)$ to $(0,1)$ with inverse

$$x = g^{-1}(y) = \frac{q}{p}\frac{y}{1-y} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{q}{p}\frac{1}{(1-y)^2}.$$

Notice that $1 + \frac{p}{q}x = 1 + \frac{y}{1-y} = \frac{1}{1-y}$. By the change of variables formula, with $C = \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$,

$$\begin{aligned} f_Y(y) &= f_X\big(g^{-1}(y)\big)\frac{dx}{dy} = C\Big(\frac{p}{q}\Big)^{\frac{p}{2}}\left(\frac{q}{p}\frac{y}{1-y}\right)^{\frac{p}{2}-1}(1-y)^{\frac{p+q}{2}} \cdot \frac{q}{p}\frac{1}{(1-y)^2} \\ &= C\Big(\frac{p}{q}\Big)^{\frac{p}{2}}\Big(\frac{q}{p}\Big)^{\frac{p}{2}} y^{\frac{p}{2}-1}(1-y)^{-\frac{p}{2}+1+\frac{p+q}{2}-2} = \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} y^{\frac{p}{2}-1}(1-y)^{\frac{q}{2}-1} \end{aligned}$$

for $0 < y < 1$, which is the $\mathrm{Beta}(\frac{p}{2}, \frac{q}{2})$ PDF.