1 Numerical Approximations

1.1 Approximating Values of a Function (Linear Approximations)

We can approximate the value of a function $f(x)$ at points near $(a, f(a))$ using a linear approximation.

Linear Approximation: The *linear approximation* of a function at $x = a$ is given by

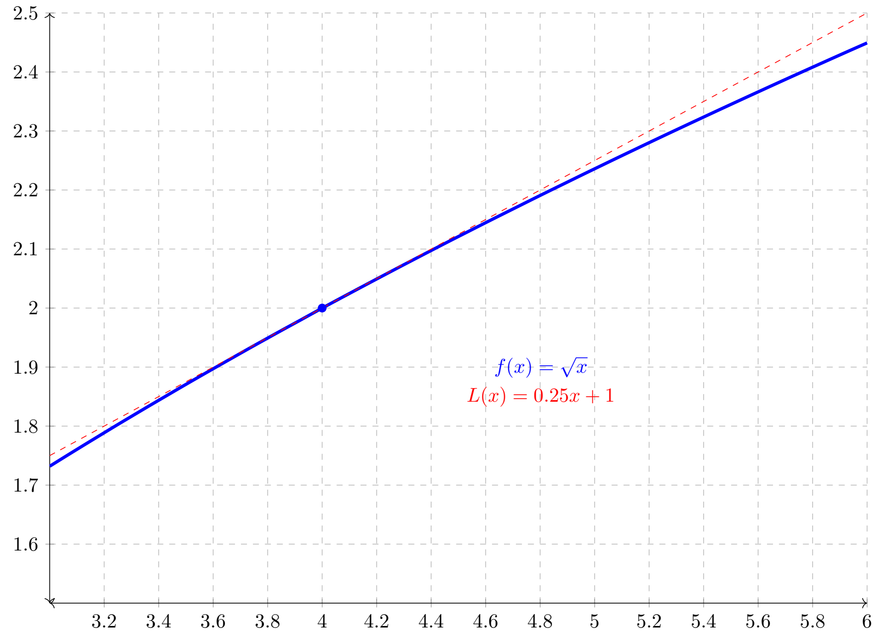
$$L(x) = f(a) + f'(a)(x - a).$$

For x close to a, we have

$$L(x) \approx f(x).$$

Remark: Notice that $L(x)$ is simply the tangent line of $f(x)$ at a.

Example 1: The linear approximation to the function $f(x) = \sqrt{x}$ at $x = 4$ is given by $L(x) = 0.25x+1$. Notice that at points near $x = 4$, the blue and red lines are very close.



1.2 Approximating Roots of a Function (Newton's Method)

Let x^* denote a root of a function f, that is, a number such that $f(x^*) = 0$. We can approximate roots of the function $f(x)$ using Newton's method.

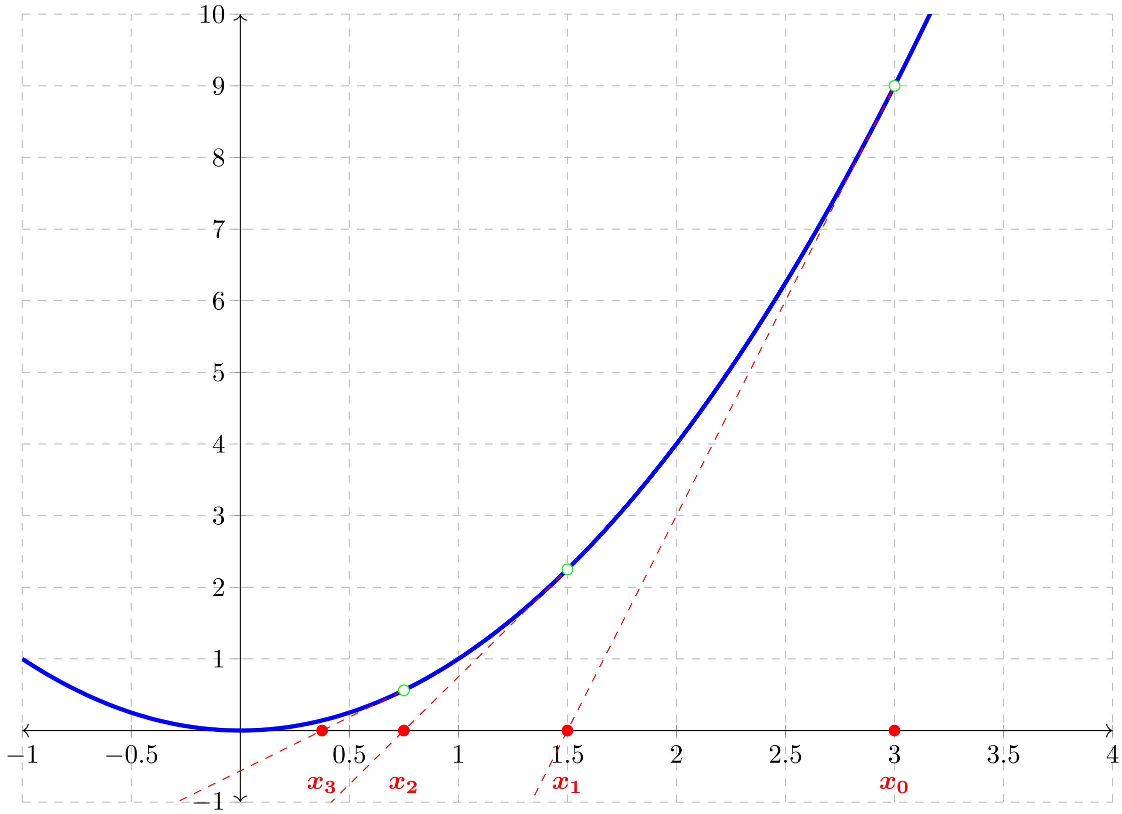
Newton's Method: *Newton's Method* gives a sequence of numbers $x_1, x_2, \ldots$ that get closer and closer to a root of $f(x)$. The sequence starting at $x_0 = a$ is defined recursively for $n \geq 0$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For large n, we have

$$x_n \approx x^*.$$

Warning: Newton's method may not approximate a root to a function if our initialization point $x_1 = a$ is chosen poorly. For example, Newton's method may fail to approximate a root if a is chosen near a local maximum or minimum.



Example 2: Newton's Method to find the roots of $f(x) = x^2$ is demonstrated in the picture above:

1. Starting from the point $x_0 = 3$, we compute the tangent line to the curve at $x = 3$. The x-intercept of the linear approximation is 1.5, which we denote by x_1.
2. Starting from the point $x_1 = 1.5$, we compute the tangent line to the curve at $x = 1.5$. The x-intercept of the linear approximation is 0.75, which we denote by x_2.
3. Starting from the point $x_2 = 0.75$, we compute the tangent line to the curve at $x = 0.75$. The x-intercept of the linear approximation is 0.375, which we denote by x_3.
4. Repeat...

The sequence of red dots x_0, x_1, x_2, x_3 on the x axis get closer and closer to the root $x^* = 0$.

1.3 Example Problems

1.3.1 Linear Approximations

Strategy: To approximate the value of a function at $x = b$, we compute the tangent line at a point a near b. The point a is chosen so that $f(a)$ and $f'(a)$ are easy to compute.

Problem 1. ($\star$) Find the linear approximation of $\ln(x)$ at $x = 1$. Estimate the value of $\ln(1.01)$.

Solution 1. We simply plug $a = 1$ into the formula of the linear approximation,

$$L(x) = f(a) + f'(a)(x - a).$$

Notice that $f(1) = \ln(1) = 0$ and $f'(1) = \frac{1}{x}\big|_{x=1} = 1$. Therefore, we have

$$L(x) = f(1) + f'(1)(x-1) = x - 1.$$

In particular, we have

$$L(1.01) = 1.01 - 1 = 0.1 \approx \ln(1.01) = 0.00995033.$$

Remark: Notice that $L(1.01)$ is an over approximation of $\ln(1.01)$. This is because $\ln(x)$ is concave down at 1, so the tangent line lies above our curve at this point.

1.3.2 Newton's Method

Strategy: To approximate the roots of a function, we usually use the intermediate value theorem to find a value "close" to our root. We then start using Newton's method for a few iterations to estimate it.

Problem 1. ($\star\star$) Let $g(x) = 3x^{1/3} - x$. Estimate the three solutions to the equation $g(x) = 1$ correct to 2 decimal places.

Solution 1. Finding solutions to $g(x) = 1$ is equivalent to finding roots of $f(x) = 3x^{1/3} - x - 1 = 0$.

Finding Values Near Roots: Since $3x^{1/3} - x - 1 = 0$ is continuous on $\mathbb{R}$, we can use the intermediate value theorem to estimate the locations of our roots. We check intervals of the form $[n, n+1]$. After several attempts, we can conclude that

1. $[0, 1]$: Since $f(0) = -1$ and $f(1) = 1$, the intermediate value theorem implies there exists a root in the interval $(0, 1)$.
2. $[3, 4]$: Since $f(3) = 0.327$ and $f(4) = -0.238$, the intermediate value theorem implies there exists a root in the interval $(3, 4)$.
3. $[-7, -6]$: Since $f(-7) = 0.261$ and $f(-6) = -0.451$, the intermediate value theorem implies there exists a root in the interval $(-7, -6)$.

Apply Newton's Method: Recall that the recursive formula for Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n^{1/3} - x_n - 1}{x_n^{-2/3} - 1}.$$

We start by approximating each of our 3 roots in the intervals we discovered above,

1. $[0, 1]$: We take $x_0 = 0.1$. Notice that

$$x_1 = 0.1 - \frac{3 \cdot 0.1^{1/3} - 0.1 - 1}{0.1^{-2/3} - 1} \approx 0.0197$$
$$x_2 = 0.0197 - \frac{3 \cdot 0.0197^{1/3} - 0.0197 - 1}{0.0197^{-2/3} - 1} \approx 0.0362$$
$$x_3 = 0.0362 - \frac{3 \cdot 0.0362^{1/3} - 0.0362 - 1}{0.0362^{-2/3} - 1} \approx 0.0416$$
$$x_4 = 0.0416 - \frac{3 \cdot 0.0416^{1/3} - 0.0416 - 1}{0.0416^{-2/3} - 1} \approx 0.0419.$$

We can conclude that the approximate value of a root in $[0, 1]$ is approximately 0.04.

2. $[3,4]$: We take $x_0 = 3$. Notice that

$$
\begin{aligned}
x_1 &= 3 - \frac{3 \cdot 3^{1/3} - 3 - 1}{3^{-2/3} - 1} \approx 3.629 \\
x_2 &= 3.629 - \frac{3 \cdot 3.629^{1/3} - 3.629 - 1}{3.629^{-2/3} - 1} \approx 3.596 \\
x_3 &= 3.596 - \frac{3 \cdot 3.596^{1/3} - 3.596 - 1}{3.596^{-2/3} - 1} \approx 3.596.
\end{aligned}
$$

We can conclude that the approximate value of a root in $[3,4]$ is approximately 3.60.

3. $[-7,-6]$: We take $x_0 = -6$. Notice that

$$
\begin{aligned}
x_1 &= -6 - \frac{-3 \cdot 6^{1/3} + 6 - 1}{-6^{-2/3} - 1} \approx -6.346 \\
x_2 &= -6.346 - \frac{-3 \cdot 6.346^{1/3} + 6.346 - 1}{-6.346^{-2/3} - 1} \approx -6.507 \\
x_3 &= -6.507 - \frac{-3 \cdot 6.507^{1/3} + 6.507 - 1}{-6.507^{-2/3} - 1} \approx -6.580 \\
x_4 &= -6.58 - \frac{-3 \cdot 6.58^{1/3} + 6.58 - 1}{-6.58^{-2/3} - 1} \approx -6.610 \\
x_5 &= -6.61 - \frac{-3 \cdot 6.61^{1/3} + 6.61 - 1}{-6.61^{-2/3} - 1} \approx -6.626 \\
x_6 &= -6.626 - \frac{-3 \cdot 6.626^{1/3} + 6.626 - 1}{-6.626^{-2/3} - 1} \approx -6.632 \\
x_7 &= -6.632 - \frac{-3 \cdot 6.632^{1/3} + 6.632 - 1}{-6.632^{-2/3} - 1} \approx -6.635 \\
x_8 &= -6.635 - \frac{-3 \cdot 6.635^{1/3} + 6.635 - 1}{-6.635^{-2/3} - 1} \approx -6.636.
\end{aligned}
$$

We can conclude that the approximate value of a root in $[-7,-6]$ is approximately -6.64.

Remark: If we tried $x_0 = 0$, then our solution would not have converged to the root in $[0,1]$, so we had to try a number slightly larger than 0. The rate of convergence of the case $x_0 = -6$ was a lot slower than the other cases.

2 L'Hôpital's Rule

We can use derivatives to compute limits that were previously impossible to solve using algebraic manipulations.

L'Hôpital's Rule: For a limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or is infinite.

Remark: L'Hôpital's Rule also works for one-sided limits and limits at infinity.

Warnings:

1. L'Hôpital's Rule does not always work. Sometimes limits need to be computed using the methods we used in week 3. If the limit computed using L'Hôpital's Rule does not exist, it does not necessarily mean that our original limit does not exist.
2. Make sure the limit is an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before attempting to apply L'Hôpital's Rule.
3. Do not confuse L'Hôpital's Rule with the quotient rule.

2.1 Example Problems

Strategy: We can only apply L'Hôpital's rule if we have the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

1. Check if our limit is an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
2. If our limit is of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can use algebra to write our limit in the appropriate form.
3. If our limit is of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we can differentiate the numerator and denominator and use the fact
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
4. Simplify our expression after differentiating the numerator and denominator, and compute the limit if possible. If not, then repeat the process until we can easily deduce our limit.

2.1.1 Indeterminate forms of the type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$

Problem 1. ($\star$) Determine the following limit

$$\lim_{x\to 1} \frac{x^7 - 1}{x^3 - 1}.$$

Solution 1. We first try to substitute $x = 1$ into our function

$$\frac{x^7 - 1}{x^3 - 1}\Big|_{x=1} = \frac{1^7 - 1}{1^3 - 1} = \frac{0}{0}.$$

This is an indeterminate form of type $\frac{0}{0}$ so we can apply L'Hôpital's Rule,

$$\begin{aligned} \lim_{x\to 1} \frac{x^7 - 1}{x^3 - 1} &= \lim_{x\to 1} \frac{7x^6}{3x^2} \qquad \text{L'Hôpital's Rule for } \frac{0}{0} \\ &= \frac{7}{3}. \end{aligned}$$

Problem 2. ($\star\star$) Determine the following limit

$$\lim_{x\to\infty} \frac{e^x + x}{x^2 + 2x + 9}.$$

Solution 2. We first try to substitute $x = \infty$ into our function

$$\frac{e^x + x}{x^2 + 2x + 9}\Big|_{x=\infty} = \frac{\infty}{\infty}.$$

This is an indeterminate form of type $\frac{\infty}{\infty}$ so we can apply L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^x + x}{x^2 + 2x + 9} &= \lim_{x\to\infty} \frac{e^x + 1}{2x + 2} && \text{L'Hôpital's Rule for } \frac{\infty}{\infty} \\
&= \lim_{x\to\infty} \frac{e^x}{2} && \text{L'Hôpital's Rule for } \frac{\infty}{\infty} \\
&= \infty.
\end{aligned}$$

Remark: In this example, we had to apply L'Hôpital's Rule twice, because one application still left us with an indeterminate form.

2.1.2 Indeterminate forms of the type $\infty - \infty$

Strategy: If our limit is of the form $\infty - \infty$, then we usually simplify our expression by either rationalizing, using a common denominator, or using basic properties of functions.

Problem 1. ($\star\star$) Determine the following limit

$$\lim_{x\to 1} \Big(\frac{1}{x-1} - \frac{1}{\ln(x)}\Big).$$

Solution 1. We first try to substitute $x = 1^+$ and $x = 1^-$ into our function

$$\Big(\frac{1}{x-1} - \frac{1}{\ln(x)}\Big)\Big|_{x=1^+} = \infty - \infty \qquad \text{and} \qquad \Big(\frac{1}{x-1} - \frac{1}{\ln(x)}\Big)\Big|_{x=1^-} = -\infty + \infty.$$

This is an indeterminate form, but it is not of the type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. We write the difference with a common denominator to simplify this expression,

$$\lim_{x\to 1} \Big(\frac{1}{x-1} - \frac{1}{\ln(x)}\Big) = \lim_{x\to 1} \Big(\frac{\ln(x) - x + 1}{x\ln x - \ln x}\Big).$$

This is an indeterminate form of type $\frac{0}{0}$ so we can apply L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to 1} \Big(\frac{\ln(x) - x + 1}{x\ln x - x}\Big) &= \lim_{x\to 1} \Big(\frac{\frac{1}{x} - 1}{\ln x + \frac{x}{x} - \frac{1}{x}}\Big) && \text{L'Hôpital's Rule for } \frac{\infty}{\infty} \\
&= \lim_{x\to 1} \Big(\frac{\frac{1}{x} - 1}{\ln x + 1 - \frac{1}{x}}\Big) \\
&= \lim_{x\to 1} \Big(\frac{-\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}}\Big) && \text{L'Hôpital's Rule for } \frac{0}{0} \\
&= -\frac{1}{2}.
\end{aligned}$$

Problem 2. ($\star\star\star$) Determine the following limit

$$\lim_{x\to\infty}\Big(\frac{1}{2}\ln x - \ln(x-1) + \frac{1}{2}\ln(x-2)\Big).$$

Solution 2. We first try to substitute $x=\infty$ into our function

$$\Big(\frac{1}{2}\ln x - \ln(x-1) + \frac{1}{2}\ln(x-2)\Big)\Big|_{x=\infty} = \infty - \infty + \infty.$$

This is an indeterminate form, but it is not of the type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. We use the properties of the logarithm to simplify this expression,

$$\lim_{x\to\infty}\frac{1}{2}\ln x - \ln(x-1) + \frac{1}{2}\ln(x-2) = \lim_{x\to\infty}\ln\frac{x^{1/2}(x-2)^{1/2}}{x-1} = \ln\Big(\lim_{x\to\infty}\frac{x^{1/2}(x-2)^{1/2}}{x-1}\Big)$$

by continuity of $\ln(x)$. The limit on the inside is of the form $\frac{\infty}{\infty}$, so we could try to use L'Hôpital's rule to evaluate this limit, but we will quickly see that this approach will not lead anywhere. Instead, we can factor out the exponent,

$$\ln\Big(\lim_{x\to\infty}\frac{x^{1/2}(x-2)^{1/2}}{x-1}\Big) = \ln\bigg(\lim_{x\to\infty}\Big(\frac{x(x-2)}{(x-1)^2}\Big)^{\frac{1}{2}}\bigg) = \frac{1}{2}\ln\Big(\lim_{x\to\infty}\frac{x(x-2)}{(x-1)^2}\Big)$$

by continuity $x^{1/2}$. The limit on the inside is of the form $\frac{\infty}{\infty}$, so we could try to use L'Hôpital's rule to evaluate this limit,

$$\begin{aligned}\frac{1}{2}\ln\Big(\lim_{x\to\infty}\frac{x(x-2)}{(x-1)^2}\Big) &= \frac{1}{2}\ln\Big(\lim_{x\to\infty}\frac{x^2-2x}{x^2-2x+1}\Big)\\ &= \frac{1}{2}\ln\Big(\lim_{x\to\infty}\frac{2x-2}{2x-2}\Big) && \text{L'Hôpital's Rule for } \frac{\infty}{\infty}\\ &= \frac{1}{2}\ln\Big(\lim_{x\to\infty}\frac{2}{2}\Big) && \text{L'Hôpital's Rule for } \frac{\infty}{\infty}\\ &= \frac{1}{2}\ln(1)\\ &= 0.\end{aligned}$$

Remark: The easiest way to compute the limit is to actually use the techniques from week 3,

$$\ln\Big(\lim_{x\to\infty}\frac{x^{1/2}(x-2)^{1/2}}{x-1}\Big) = \ln\Big(\lim_{x\to\infty}\frac{x^{1/2}(x-2)^{1/2}}{x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\Big) = \ln\Big(\lim_{x\to\infty}\frac{(1-\frac{2}{\sqrt{x}})^{1/2}}{1-\frac{1}{x}}\Big) = \ln(1) = 0.$$

2.1.3 Indeterminate forms of the type $0\cdot\infty$

Strategy: If our limit is of the form $0\cdot\infty$, then we use the fact that

$$f(x)g(x) = \frac{g(x)}{f(x)^{-1}} = \frac{f(x)}{g(x)^{-1}}$$

to simplify our limit. If L'Hôpital's Rule fails when using the simplification $f(x)g(x) = \frac{g(x)}{f(x)^{-1}}$, then usually trying the simplification $f(x)g(x) = \frac{f(x)}{g(x)^{-1}}$ will work.

Problem 1. ($\star$) Determine the following limit

$$\lim_{x\to 0^+} x\ln(x).$$

Solution 1. We first try to substitute $x = 0^+$ into our function

$$x \ln(x)\Big|_{x=0^+} = 0 \cdot -\infty.$$

This is an indeterminate form, but it is not of the type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. We simplify our expression first,

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}}$$

This is an indeterminate form of type $\frac{\infty}{\infty}$ so we can apply L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} && \text{L'Hôpital's Rule for } \frac{\infty}{\infty} \\ &= \lim_{x\to 0^+} -x \\ &= 0.\end{aligned}$$

Remark: If we tried to compute the limit

$$\lim_{x\to 0} x\ln(x) = \lim_{x\to 0} \frac{x}{\ln(x)^{-1}},$$

then the procedure would not have terminated.

2.1.4 Indeterminate forms of the type 0^0, ∞^0, 1^∞

Strategy: If our limit is of the form 0^0, ∞^0, 1^∞, then we use the fact that

$$f(x)^{g(x)} = e^{\ln(f(x)^{g(x)})} = e^{g(x)\cdot \ln f(x)}$$

to simplify our limit.

Problem 1. ($\star\star$) Determine the following limit

$$\lim_{x\to 0} (\sec(x))^{\frac{1}{x^2}}.$$

Solution 1. We first try to substitute $x = 0$ into our function

$$(\sec(x))^{\frac{1}{x^2}}\Big|_{x=0} = 1^\infty.$$

This is an indeterminate form, but it is not of the type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. We simplify our expression first,

$$\lim_{x\to 0} (\sec(x))^{\frac{1}{x^2}} = \lim_{x\to 0} \exp\left(\ln\left((\sec(x))^{\frac{1}{x^2}}\right)\right) = \exp\left(\lim_{x\to 0}\left(\frac{\ln\sec(x)}{x^2}\right)\right)$$

by continuity of e^x. The limit on the inside is of the form $\frac{0}{0}$, so we could try to use L'Hôpital's rule to evaluate this limit,

$$\begin{aligned}\exp\left(\lim_{x\to 0}\left(\frac{\ln\sec(x)}{x^2}\right)\right) &= \exp\left(\lim_{x\to 0}\left(\frac{\frac{1}{\sec(x)}\sec(x)\tan(x)}{2x}\right)\right) && \text{L'Hôpital's Rule for } \frac{0}{0} \\ &= \exp\left(\lim_{x\to 0}\left(\frac{\sec^2(x)}{2}\right)\right) && \text{L'Hôpital's Rule for } \frac{0}{0} \\ &= \exp(1/2).\end{aligned}$$

Problem 2. ($\star\star$) Determine the following limit

$$\lim_{x\to 0^+} 2x^{x^2}.$$

Solution 2. We first try to substitute $x = 0^+$ into our function

$$2x^{x^2}\Big|_{x=0^+} = 2\cdot 0^0.$$

This is an indeterminate form, but it is not of the type $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. We simplify our expression first,

$$\lim_{x\to 0^+} 2x^{x^2} = \lim_{x\to 0^+} 2\exp\Big(\ln\big(x^{x^2}\big)\Big) = 2\exp\Big(\lim_{x\to 0^+}\big(x^2\ln x\big)\Big)$$

by continuity of e^x. The limit on the inside is of the form $0\cdot\infty$, so we have to do one more simplification before we can use L'Hôpital's rule to evaluate this limit,

$$\begin{aligned}
2\exp\left(\lim_{x\to 0^+}\left(x^2\ln x\right)\right) &= 2\exp\left(\lim_{x\to 0^+}\left(\frac{\ln x}{x^{-2}}\right)\right)\\
&= 2\exp\left(\lim_{x\to 0^+}\left(\frac{x^{-1}}{-2x^{-3}}\right)\right) && \text{L'Hôpital's Rule for } \frac{0}{0}\\
&= 2\exp\left(\lim_{x\to 0^+}\left(\frac{x^2}{-2}\right)\right)\\
&= 2.
\end{aligned}$$

2.1.5 When L'Hôpital's Rule Fails

Strategy: L'Hôpital's Rule is a powerful tool that allows us to compute many limits we could not solve before. However, it is still important to remember our methods in week 3, because not all limits can be solved using L'Hôpital's Rule.

Problem 1. ($\star$) Determine the following limit

$$\lim_{x\to 4}\frac{x^2+x-4}{x+4}.$$

Solution 1. We first try to substitute $x = 4$ into our function

$$\frac{x^2+x-4}{x+4}\Big|_{x=4} = \frac{4^2+4-4}{4+4} = 2.$$

This is not an indeterminate form, so we can conclude

$$\lim_{x\to 4}\frac{x^2+x-4}{x+4} = 2.$$

Remark: We always have to check if our limit is in the appropriate form before applying L'Hôpital's Rule. For example, if we blindly applied the rule, then we would have incorrectly concluded

$$\lim_{x\to 4}\frac{x^2+x-4}{x+4} = \lim_{x\to 4}\frac{2x+1}{1} = 9.$$

Problem 2. ($\star\star$) Determine the following limit

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x}.$$

Solution 2. We first try to substitute $x = \infty$ into our function

$$\frac{x+\sin(x)}{x}\Big|_{x=\infty} = \frac{\infty}{\infty}.$$

This is an indeterminate form of type $\frac{\infty}{\infty}$ so we can try to apply L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x} = \lim_{x\to\infty} \frac{1+\cos(x)}{1}.$$

The limit on the right does not exist, so L'Hôpital's Rule actually cannot be applied in this step. We might be tempted to conclude that this means our limit does not exist as well, but this is wrong.

In fact, using our methods from week 3, the limit actually exists

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x} = \lim_{x\to\infty} 1 + \frac{\sin(x)}{x} = 1,$$

by the squeeze theorem.