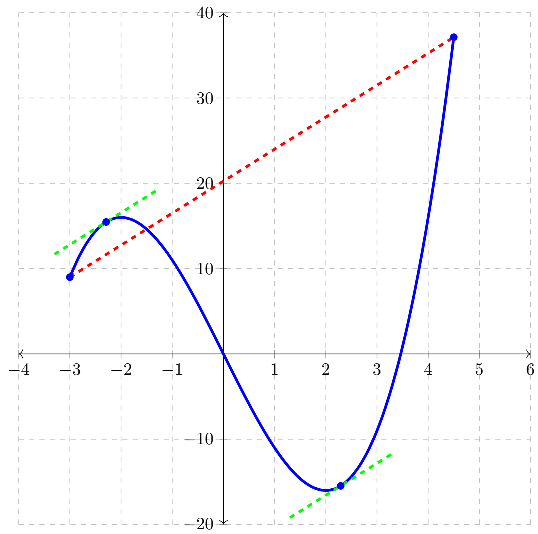
1 Mean Value Theorem

Theorem 1 (Mean Value Theorem)**.** *Let f be a continuous on $[a, b]$ and differentiable on (a, b). There exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 1: The Mean Value Theorem says there exists at least one point $c \in (a, b)$ such that the tangent line to the curve at $(c, f(c))$ is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.



Remark: If we also assume that $f(a) = f(b)$, then the mean value theorem says there exists a $c \in [a, b]$ such that $f'(c) = 0$. This result is called Rolle's Theorem.

1.1 Consequences of the Mean Value Theorem

Corollary 1. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on the interval (a, b).*

Corollary 2. *If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there is a constant C such that for all $x \in (a, b)$,*

$$f(x) = g(x) + C.$$

Corollary 3. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b).*

Corollary 4. *If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).*

1.2 Example Problems

Problem 1. $(\star\star\star)$ Show that $|\sin(x)| < x$ for all $x > 0$.

Solution 1. Consider the function $f(x) = |\sin(x)| - x$. We consider the two cases.

1. $0 < x < 1$: We have $\sin(x) > 0$, so $f'(x) = \cos(x) - 1 < 0$ for all $x \in (0, 1)$. Corollary 4 implies f is strictly decreasing on $x \in (0, 1)$. Therefore, for $x \in (0, 1)$, we have

$$0 = f(0) > f(x) = \sin(x) - x \implies \sin(x) < x \text{ for } x \in (0, 1).$$

2. $x \geq 1$: Notice that $|\sin(1)| < 1$ and $|\sin(x)| \leq 1$ for all x. Therefore, we must have $|\sin(x)| < x$ for all $x \geq 1$.

Problem 2. ($\star\star$) Show that $1 + x < e^x$ for all $x > 0$.

Solution 2. Consider the function $f(x) = 1+x-e^x$. We have $e^x > 1$ for $x > 0$, so $f'(x) = 1-e^x < 0$ for all $x > 0$. Corollary 4 implies f is strictly decreasing for $x > 0$. Therefore, for $x > 0$, we have

$$0 = f(0) > f(x) = 1 + x - e^x \implies 1 + x < e^x \text{ for } x > 0.$$

Alternate Method: Let $x > 0$. Notice that the function e^t is continuous on $[0, x]$ and differentiable on $(0, x)$, so the mean value theorem states there exists a $c \in (0, x)$ such that

$$f'(c) = \frac{e^x - e^0}{x - 0} = \frac{e^x - 1}{x}.$$

Furthermore, we have $f'(c) = e^c > 1$ since $c > 0$. Therefore, the equation above implies

$$\frac{e^x - 1}{x} = f'(c) > 1 \implies e^x - 1 > x \implies e^x > 1 + x.$$

Problem 3. ($\star\star\star$) Use the Mean Value Theorem to prove Corollary 1.

Solution 3. Suppose that $f'(x) = 0$ for all $x \in (a, b)$. Consider the points $a < x_1 < x_2 < b$. Since f' exists on (a, b), we have f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore, by the Mean Value Theorem, there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, we must have $f'(c) = 0$, which implies that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \implies f(x_1) = f(x_2).$$

Since x_1 and x_2 were arbitrary, we can conclude that $f(x_1) = f(x_2)$ for all $x_1, x_2 \in (a, b)$. That is, $f(x)$ is constant.

Problem 4. ($\star\star\star$) Prove Corollary 2.

Solution 4. Suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Consider the function $h(x) = f(x) - g(x)$. By assumption we have $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$. Therefore, using Corollary 1, we can conclude that there exists a constant C such that $h(x) = C$ on (a, b). Therefore, we have

$$C = h(x) = f(x) - g(x) \implies f(x) = g(x) + C.$$

Problem 5. ($\star\star\star$) Prove Corollary 3.

Solution 5. Suppose that $f'(x) > 0$ for all $x \in (a, b)$. To show a function is strictly increasing, we need to show that

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

Consider the points $a < x_1 < x_2 < b$. Since f' exists on (a, b), we have f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore, by the Mean Value Theorem, there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, we must have $f'(c) > 0$, which implies that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \implies f(x_2) > f(x_1).$$

Remark: To prove Corollary 4, we can use the fact that f is strictly decreasing on (a, b) is equivalent to $-f$ is strictly increasing on (a, b). Corollary 4 then follows as a direct consequence of Corollary 3.

2 Extreme Values

Definition 1. The largest and smallest values a function can take on an interval I are called the *global extreme values*. There are two types:

1. f has a *global maximum* on I at $c \in I$ if $f(x) \leq f(c) = M$ for all $x \in I$. The number M is called the *global maximum value* of f. The point c is sometimes called the *global maximizer*.

2. f has a *global minimum* on I at $c \in I$ if $f(x) \geq f(c) = m$ for all $x \in I$. The number m is called the *global minimum value* of f. The point c is sometimes called the *global minimizer*.

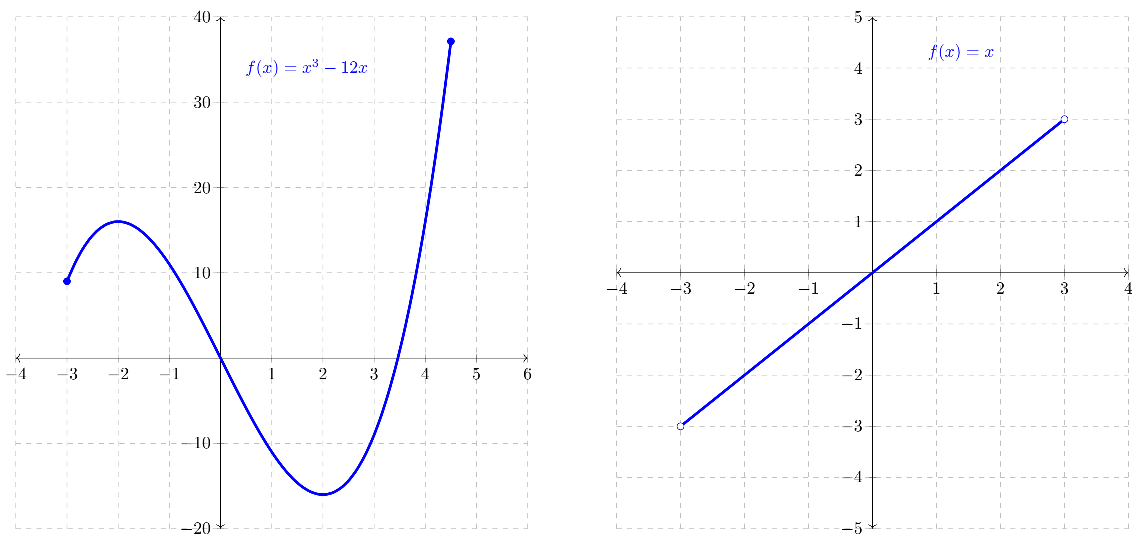
Definition 2. The largest and smallest values a function can take near a point are called *local extreme values*. There are two types:

1. f has a *local maximum* at c if there exists an open interval I containing c such that $f(x) \leq f(c)$ for all $x \in I$.

2. f has a *local minimum* at c if there exists an open interval I containing c such that $f(x) \geq f(c)$ for all $x \in I$.

Example 2: The function $f(x) = x^3 - 12x$ on the interval $[-3, 4.5]$ has a local maximum at $x = -2$ and a local minimum at $x = 2$. The function has a global maximum value of $f(4.5) = 37.125$ at $x = 4.5$ and a global minimum value of $f(2) = -16$ at $x = 2$.

Example 3: The function $f(x) = x$ on the interval $(-3, 3)$ does not have any extreme values. Even though it looks like the the global minimums and maximums of f occur at -3 and 3, $f(3)$ and $f(-3)$ do not exist since -3 and 3 are not in the interval $(-3, 3)$, so they cannot be extreme values.



2.1 Existence of Extreme Values

Some functions may not have extreme values. The following theorem provides some conditions that guarantee the existence of extreme values.

Theorem 2 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$ then f has a global maximum and a global minimum on $[a, b]$. That is, there exists points c and d in $[a, b]$ such that*

$$f(c) \leq f(x) \leq f(d) \text{ for all } x \in [a, b].$$

2.2 Location of Extreme values

Definition 3. The *critical points* of f are the values c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 3. *If $f(x)$ has a local extreme at c, then c is a critical point of f.*

This means we look for local extremes at the critical points of our function. However, this does not mean that every critical point corresponds to a extreme value. Theorem 3 is insufficient to determine if the critical point is a local minimum, local maximum, or neither.

2.3 Classification of Extreme values

We can use something called the first derivative test to classify continuous function at critical points.

Theorem 4 (First Derivative Test)**.** *Suppose that c is a critical point of a continuous function f.*

1. *If f' changes signs from negative to positive at c, then f has a local minimum at c.*
2. *If f' changes signs from positive to negative at c, then f has a local minimum at c.*

If our function is twice differentiable, we can instead use something called the second derivative test to classify critical points of nice twice differentiable functions.

Theorem 5 (Second Derivative Test)**.** *Suppose that $f''(x)$ is continuous near c.*

1. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
2. *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

To find the global minimum and maximum values of a function on an interval $[a, b]$, it suffices to evaluate our function at the critical points and the boundary of our interval and choose the corresponding smallest and largest values.

2.4 Example Problems

Problem 1. ($\star\star$) Find all local and global extreme values of $f(x) = \frac{e^x+e^{-x}}{2}$.

Solution 1.

Finding the Critical Points: We first find the critical points of our function. Using the chain rule, we see

$$f'(x) = \frac{e^x - e^{-x}}{2}.$$

The derivative exists for all $x \in \mathbb{R}$, so we can set this equal to 0 to find the remaining critical points,

$$f'(x) = \frac{e^x - e^{-x}}{2} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = -x \Rightarrow x = 0.$$

Classifying the Critical Points: To classify the critical point at $x = 0$, notice that

$$f''(0) = \frac{e^x + e^{-x}}{2}\Big|_{x=0} = 1 > 0,$$

so there is a local minimum when $x = 0$. The minimum value at this local minimum is $f(0) = 0$.

We now search for global maximum and minimum values. Taking $x \to \pm\infty$, we see that

$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{2} = \infty \quad \text{and} \quad \lim_{x\to-\infty} \frac{e^x + e^{-x}}{2} = \infty$$

so there are no global maximums. Since both positive and negative limits at infinity are ∞, this function has a global minimum, and it must coincide with the local minimum.

Remark: We classified the extreme points of $f(x) = \cosh(x)$.

Problem 2. ($\star\star$) Find the global maximum of $f(x) = |\sin(x)| - |x|$ on the interval $[-1, 1]$. Use this to conclude that

$$|\sin(x)| \leq |x| \quad \text{for } -1 \leq x \leq 1.$$

Solution 2.

Finding the Critical Points: We first find the critical points of our function. We first differentiate f for $x \neq 0$,

$$f'(x) = \begin{cases} \frac{d}{dx}(\sin(x) - x) & x \in (0,1] \\ \frac{d}{dx}(-\sin(x) + x) & x \in [-1,0) \end{cases} = \begin{cases} \cos(x) - 1 & x \in (0,1] \\ -\cos(x) + 1 & x \in [-1,0) \end{cases}$$

Notice that $\lim_{x\to 0^-} f'(x) = 0 = \lim_{x\to 0^+} f'(x)$, so $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$. Therefore, there is a critical point when $x = 0$.

We now set the derivative equal to 0 to find the remaining critical points. For $x \in (0, 1]$, we have

$$f'(x) = \cos(x) - 1 = 0 \Rightarrow \cos(x) = 1 \Rightarrow x = 2k\pi,$$

which do not lie in the interval $(0, 1]$ so there are no positive critical points. Similarly, for $x \in [-1, 0)$, we have

$$f'(x) = -\cos(x) + 1 = 0 \Rightarrow \cos(x) = 1 \Rightarrow x = 2k\pi,$$

which do not lie in the interval $[-1, 0)$ so there are no negative critical points. Therefore, the only critical point of $f(x)$ is $x = 0$.

Classifying the Critical Points: To find the global maximum, it suffices to compute $f(x)$ at its critical points or at the boundary,

$$f(0) = |\sin(0)| - |0| = 0, \qquad f(1) = |\sin(1)| - |1| \approx -0.159, \qquad f(-1) = |\sin(-1)| - |-1| \approx -0.159.$$

The largest of these values occurs when $x = 0$, so f attains a global maximum value of 0 when $x = 0$.

Since f has a global maximum at $x = 0$, for all $x \in [-1, 1]$ we have

$$0 = f(0) \geq |\sin(x)| - |x| \implies |\sin(x)| \leq |x| \quad \text{for } -1 \leq x \leq 1.$$

Remark: This problem demonstrates another way to prove some inequalities without using the mean value theorem. This inequality is identical to the one that appears Example 1.2.1 in Week 8 we proved using the mean value theorem. This type of bound was used in the squeeze theorem example in Problem 4 in Section 1.5.8 in Week 3.

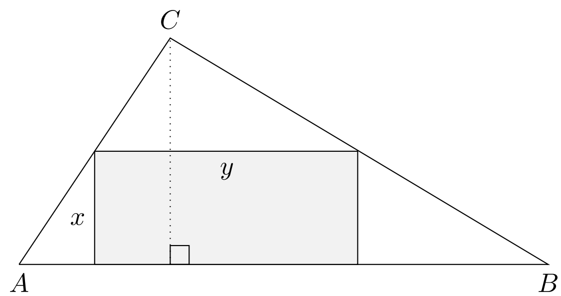
2.4.1 Optimization Problems

Strategy: Find a function of one variable that describes the quantity we want to maximize. Then find the global extreme points of this function to recover the maximum and minimum.

Problem 1. ($\star\star$) Find the area of the largest square that can be drawn in a triangle (with one of its sides along a side of the triangle).

Solution 1.

Finding the Function to Optimize: It is clear that any square that maximizes area must have its corners touching the edges of the triangle. Otherwise we can increase the area by enlarging one side of the rectangle. Consider an arbitrary triangle $\triangle ABC$ and inscribed square given attached to side $\overline{AB}$ below:



The area of the square is

$$A(x, y) = x \cdot y.$$

We need to write this as a one variable function. Let $b = \overline{AB}$ be the length of the base of $\triangle ABC$, and let h be the height of $\triangle ABC$. Notice that by similar triangles, we must have

$$\frac{b}{h} = \frac{y}{h - x} \implies y = \frac{b(h - x)}{h}.$$

Therefore, the area can be expressed in terms of x as

$$A(x) = x \cdot \frac{b(h - x)}{h} = bx - \frac{bx^2}{h} \qquad x \in [0, h].$$

Maximizing the Function: We want to maximize $A(x)$. By the Extreme Value Theorem, there exists a global maximum for this function. Since $A(x)$ is differentiable, the critical points of this function can be found by setting $A'(x) = 0$,

$$A'(x) = b - \frac{2bx}{h} = 0 \implies x = \frac{h}{2} \in [0, h].$$

Checking the value of $A(x)$ at the points $x = 0, x = h$ and $x = \frac{h}{2}$, we see

$$A(0) = 0, \qquad A(h) = 0, \qquad A\Big(\frac{h}{2}\Big) = \frac{bh}{4}.$$

Therefore, maximum area of a square inscribed in $\triangle ABC$ with edge attached to side $\overline{AB}$ is $\frac{bh}{4}$. Notice that the area of the triangle is $\frac{bh}{2}$, so the maximum area of the inscribed square is half the area of the triangle. In particular, this area does not depend on the choice of side we attached the square to, so by symmetry we can conclude that the maximum area of a square with one side along the triangle is $\frac{bh}{4}$ or half the area of the triangle.

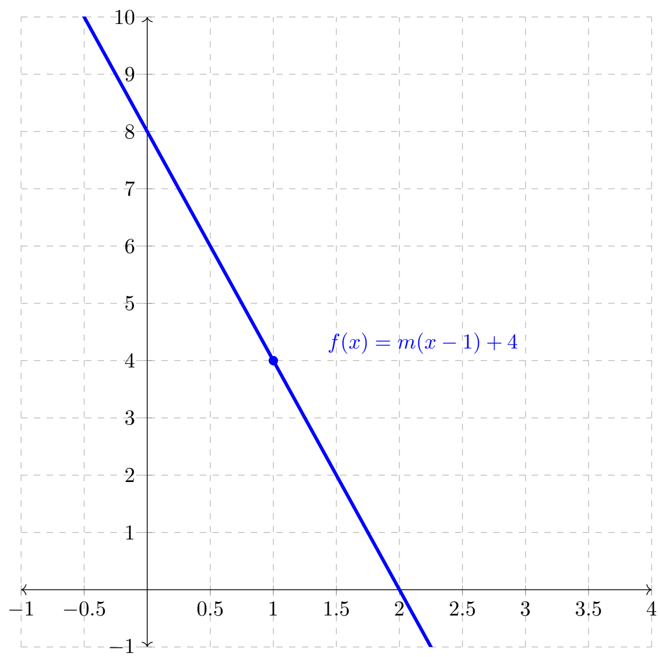
Problem 2. (⋆⋆) Find an equation of a line through the point $(1, 4)$ that cuts off the least area in the first quadrant.

Solution 2.

Finding the Function to Optimize: The equation of any line through the point $(1, 4)$ with slope m is given by

$$y = m(x - 1) + 4.$$

Clearly, if $m \geq 0$, then the area cut off will be infinite, so it suffices to consider the case where $m < 0$. Consider the following diagram:



In this case, the area cut off is given by

$$A(x, y) = \frac{1}{2} x_0 y_0,$$

where x_0 and y_0 are the corresponding x and y intercepts of the line.

We want to write the area as a function of one variable. In this case, we choose to write the area in terms of the slope of the line. For fixed m, by setting $x = 0$ and solving for y, we see

$$y_0 = m(0 - 1) + 4 \implies y_0 = -m + 4.$$

And similarly, setting $y = 0$ and solving for x, we see

$$0 = m(x_0 - 1) + 4 \implies x_0 = \frac{-4}{m} + 1.$$

Therefore, the area as a function of the slope is given by

$$A(m) = \frac{1}{2}(-m + 4) \cdot \Big(\frac{-4}{m} + 1\Big) = \frac{1}{2}\Big(-m - \frac{16}{m} + 8\Big) \quad \text{for } m < 0.$$

Minimizing the Function: We want to minimize $A(m)$. Since A is differentiable for $m < 0$, we can find the critical points by setting $A'(m) = 0$,

$$A'(m) = \frac{8}{m^2} - \frac{1}{2} = 0 \implies m = \pm 4.$$

We conclude that the only critical point occurs when $m = -4$, since $m = 4$ is not a point in our domain. To classify this critical point, notice that,

$$A''(4) = -\frac{16}{m^3}\Big|_{m=-4} = \frac{1}{4} > 0,$$

so the $A(m)$ has a local minimum at 4. Checking our function as m tends to the boundaries, we notice

$$\lim_{m \to 0^+} A(m) = \infty \quad \text{and} \quad \lim_{m \to -\infty} A(m) = \infty,$$

so both of these limits are infinity. As a consequence of the extreme value theorem (after truncating the open interval), we can conclude the global minimum exists since $A(m)$ is continuous on its domain. This global minimum must coincide with the local minimum since there is only one critical point, so we have $A(m)$ is minimized when $m = -4$. Therefore, the equation of the line that minimizes area is given by

$$y = -4(x - 1) + 4 = 8 - 4x.$$

Remark: We can also show that the local minimum is a global minimum by examining the first derivative. Since $A'(m)$ is continuous for $m < 0$, it is easy to verify that

$$A'(m) > 0 \text{ for } -4 < m < 0 \qquad \text{and} \qquad A'(m) < 0 \text{ for } -\infty < m < -4.$$

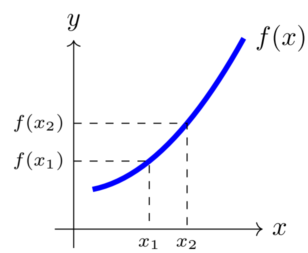
Therefore, $A(m) > A(-4)$ for $-4 < m < 0$ and $A(m) > A(-4)$ for all $m < -4$, so $A(-4)$ is a global minimum.

Remark: Yet another way to show that the local minimum is a global minimum is by examining the second derivative. Since $A''(m) > 0$ for all $m < 0$, we can conclude that $A(m)$ is concave up for $m < 0$. Therefore, the curve lies above its tangent lines, so $A(m) > A(-4)$, since $y = A(-4)$ is a horizontal tangent line.

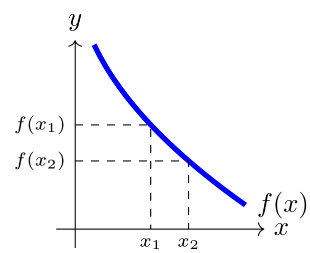
3 Curve Sketching

We break down the general steps to sketch curves of twice differentiable functions

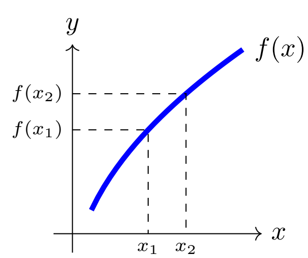
1. Optional: Find the domain of our function and any symmetries.
2. Find the location of the asymptotes (vertical, slant, horizontal).
3. Classify the signs of $f(x)$:
   (a) If $f(x) > 0$ on an interval I, then we know $f(x) > 0$ for all $x \in I$.
   (b) If $f(x) < 0$ on an interval I, then we know $f(x) < 0$ for all $x \in I$.
   (c) If $f(x) = 0$ then x is an x intercept.
4. Classify the signs of $f'(x)$:
   (a) If $f'(x) > 0$ on an interval I, then we know $f(x) > 0$ is increasing on I.
   (b) If $f'(x) < 0$ on an interval I, then we know $f(x) < 0$ is decreasing on I
   (c) If $f'(x) = 0$ then x may be a local minimum or maximum. We can apply the first derivative test to classify the extreme point.
5. Classify the signs of $f''(x)$:
   (a) If $f''(x) > 0$ on an interval I, then we know $f(x) > 0$ is concave up on I.
   (b) If $f''(x) < 0$ on an interval I, then we know $f(x) < 0$ is concave down on I.
   (c) If $f''(x) = 0$ then x may be an inflection point. If f'' changes sign at x, then there is an inflection point at $(x, f(x))$.
6. Sketch the curve incorporating all the above pieces of information. The curve should include the local extreme points, intercepts, and inflection points.



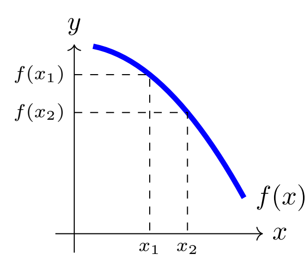
f is increasing and f is concave up



f is decreasing and f is concave up



f is increasing and f is concave down



f is decreasing and f is concave down

Figure 1: The plots above show how concavity affects the shapes of increasing and decreasing functions. A function that is concave up lies above its tangent lines, and a function that is concave down lies below its tangent lines.

3.1 Example Problems

Problem 1. ($\star\star$) Sketch the curve of $f(x) = e^{\frac{1}{x^2-1}}$ given that

$$f'(x) = \frac{-2x}{(x^2-1)^2} e^{\frac{1}{x^2-1}}, \qquad f''(x) = \frac{6x^4-2}{(x^2-1)^4} e^{\frac{1}{x^2-1}}.$$

Solution 1.

1. The domain of $f(x)$ is $\mathbb{R} \setminus \{-1, 1\}$. Also notice that f is even because

$$f(x) = e^{\frac{1}{x^2-1}} = e^{\frac{1}{(-x)^2-1}} = f(-x).$$

This means that our curve should be symmetric about the y-axis.

2. We now find the asymptototes of our graph:

Horizontal Asymptotes: Notice that

$$\lim_{x\to\infty} e^{\frac{1}{x^2-1}} = 1 \quad \text{and} \quad \lim_{x\to-\infty} e^{\frac{1}{x^2-1}} = 1,$$

so there is a horizontal asymptote of 1 at ∞ and a horizontal asymptote of 1 at $-\infty$.

Vertical Asymptotes: We look for vertical asymptotes where are function is undefined. Notice that

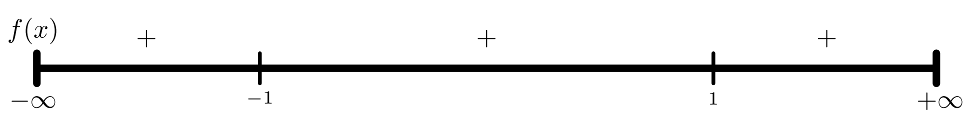
$$\lim_{x\to1^-} e^{\frac{1}{x^2-1}} = 0 \quad \text{and} \quad \lim_{x\to1^+} e^{\frac{1}{x^2-1}} = \infty$$

so there is a vertical asymptote at 1. Similarly,

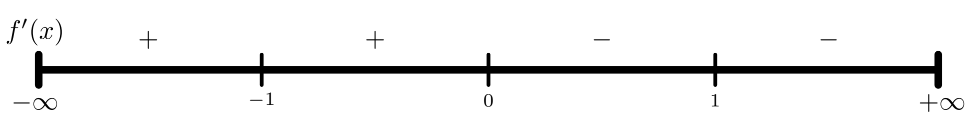
$$\lim_{x\to-1^-} e^{\frac{1}{x^2-1}} = \infty \quad \text{and} \quad \lim_{x\to-1^+} e^{\frac{1}{x^2-1}} = 0$$

so there is a vertical asymptote at -1.

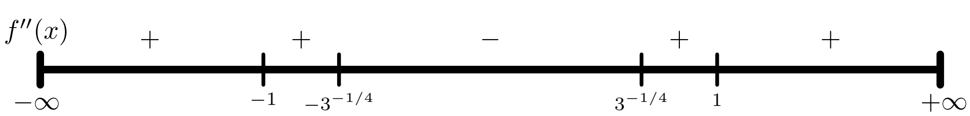
3 - 5. We now classify the signs of f, f' and f''. See Week 4 Section 1.2.2 Problem 2 for the general procedure. The signs are summarized below:



This says that the curve is always positive in its domain.

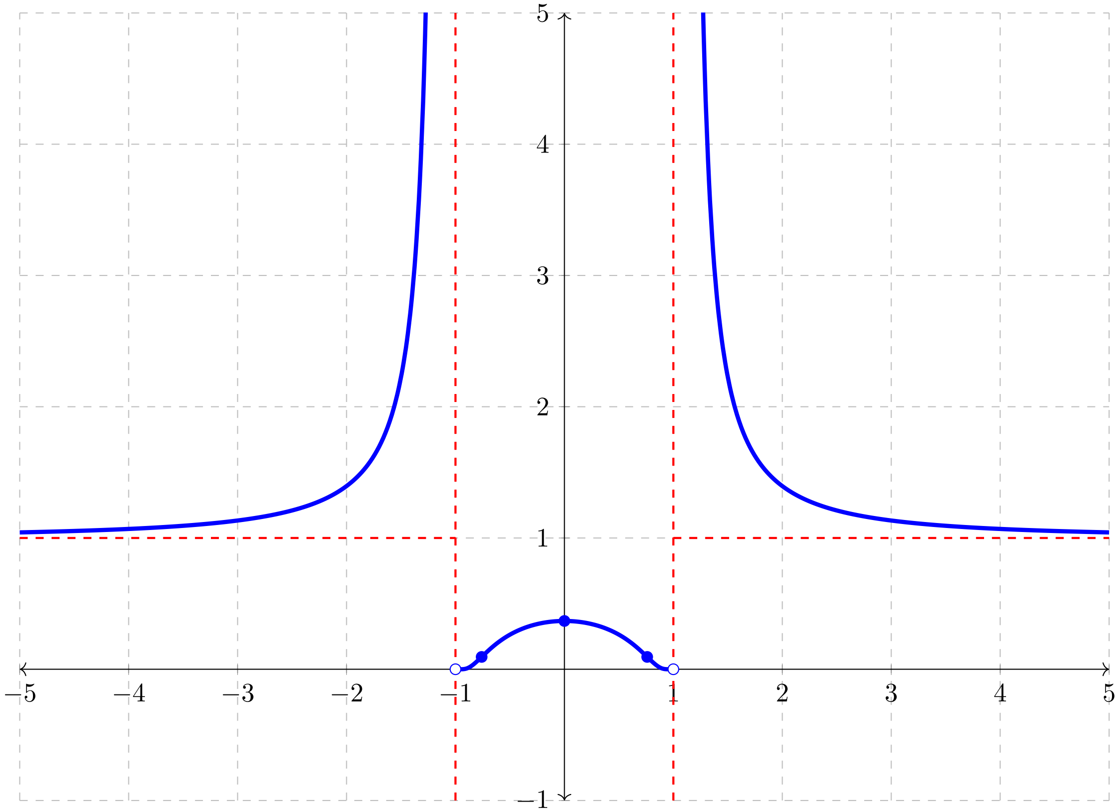


The sign of f' changes from positive to negative at $x = 0$, so there is a local maximum of $f(0) = e^{-1}$ when $x = 0$.



The sign of f'' changes at $\pm 3^{-1/4}$, so there are inflection points when $x = \pm 3^{-1/4}$. The value of f at these points are $f(3^{-1/4}) \approx 0.094$ and $f(-3^{-1/4}) \approx 0.094$

6. The curve is plotted below:



The asymptotes are the dotted red lines. The local maximum and the inflection points are marked with the blue dots. Our function is not defined when $x = \pm 1$, so there are hollow dots there.

Remark: We could have used the fact that $f(x)$ is even and draw half the curve for $x \geq 0$ and reflect along the y-axis to recover the missing half.