1 Indefinite Integrals

Definition 1. The *antiderivative* of $f(x)$ on $[a,b]$ is a function $F(x)$ such that $F'(x) = f(x)$ for all $x \in [a,b]$. The antiderivative is not unique since for any antiderivative $F(x)$ of $f(x)$, $F(x) + C$ is also an antiderivative for all integration constants $C \in \mathbb{R}$. We will use the notation

$$\int f(x)\,dx = F(x) + C$$

called the *indefinite integral* of f to denote the family of all antiderivatives of f.

1.1 Table of Integrals (omitting the integration constant)

Elementary Functions

$$\int x^a\,dx = \frac{1}{a+1}\cdot x^{a+1}, \quad a \neq -1 \tag{1}$$

$$\int \frac{1}{x}\,dx = \ln|x| \tag{2}$$

Exponential and Logarithms

$$\int e^x\,dx = e^x \tag{3}$$

$$\int \ln(x)\,dx = x\ln(x) - x \tag{4}$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} \tag{5}$$

$$\int \log_a(x)\,dx = \frac{x\ln(x) - x}{\ln(a)} \tag{6}$$

Trigonometric Functions

$$\int \sin(x)\,dx = -\cos(x) \tag{7}$$

$$\int \cos(x)\,dx = \sin(x) \tag{8}$$

$$\int \tan(x)\,dx = -\ln|\cos(x)| \tag{9}$$

$$\int \cot(x)\,dx = \ln|\sin(x)| \tag{10}$$

$$\int \sec(x)\,dx = \ln|\tan(x) + \sec(x)| \tag{11}$$

$$\int \csc(x)\,dx = -\ln|\cot(x) + \csc(x)| \tag{12}$$

Rational Functions

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) \tag{13}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) \tag{14}$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(|x|) \tag{15}$$

Hyperbolic Functions

$$\int \sinh(x)\,dx = \cosh(x) \tag{16}$$

$$\int \cosh(x)\,dx = \sinh(x) \tag{17}$$

$$\int \tanh(x)\,dx = \ln|\cosh(x)| \tag{18}$$

1.2 Basic Property

For constants $a, b \in \mathbb{R}$ and integrable functions f and g, we have

Linearity:

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

We will introduce more basic properties in the coming weeks.

1.3 Example Problems

1.3.1 Finding Indefinite Integrals

Problem 1.1. ($\star$) Find the indefinite integral

$$\int e^{-7x}\,dx.$$

Solution 1.1. It is easy to check that $-\frac{1}{7}\cdot e^{-7x}$ is an antiderivative of e^{-7x}. Therefore,

$$\int e^{-7x}\,dx = -\frac{1}{7}\cdot e^{-7x} + C.$$

Problem 1.2. ($\star\star$) Find the indefinite integral

$$\int \frac{x^3+8}{x}\,dx.$$

Solution 1.2. Using the linearity property, we have

$$\int \frac{x^3+8}{x}\,dx = \int x^2\,dx + 8\int \frac{1}{x}\,dx = \frac{1}{3}x^3 + 8\ln|x| + C.$$

1.3.2 Checking Antiderivatives

Strategy: To check if a function $F(x)$ is an antiderivative of f, it suffices to just differentiate F and check if $F'(x) = f(x)$.

Problem 1.3. ($\star\star$) Check that both $-\sin^{-1}(x)$ and $\cos^{-1}(x)$ are antiderivatives of

$$\frac{-1}{\sqrt{1-x^2}} \text{ on } (-1,1).$$

Show that

$$\cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2}.$$

Solution 1.3. From the table of derivatives, we have

$$\frac{d}{dx}(-\sin^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}} \text{ and } \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}.$$

so both $-\sin^{-1}(x)$ and $\cos^{-1}(x)$ are antiderivatives of $\frac{-1}{\sqrt{1-x^2}}$ on its domain. We know that all antiderivatives on an interval differ by an integration constant, that is $\cos^{-1}(x) = -\sin^{-1}(x) + C$ for $x \in (-1,1)$. To find this integration constant, we can evaluate our functions at 0,

$$\cos^{-1}(0) = \frac{\pi}{2} \text{ and } -\sin^{-1}(0) = 0 \implies \cos^{-1}(0) = -\sin^{-1}(0) + C \implies C = \frac{\pi}{2}.$$

Since the difference of any two antiderivatives is a constant, we conclude that the two antiderivatives must differ by $\frac{\pi}{2}$ for all x in the domain,

$$\cos^{-1}(x) = -\sin^{-1}(x) + \frac{\pi}{2} \implies \cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2}.$$

2 Differential Equations

2.1 Separable Differential Equations

Suppose that we want to find the antiderivative of some special implicit functions. In particular, suppose that we want to find a function y such that it satisfies the *differential equation*

$$\frac{dy}{dx} = f(x)g(y)$$

for some functions $f(x)$ and $g(y)$. We can find the solution by separating variables and integrating both sides,

$$\frac{dy}{dx} = f(x)g(y) \implies \frac{dy}{g(y)} = f(x)dx \implies \int \frac{1}{g(y)}\,dy = \int f(x)\,dx \implies G(y) = F(x) + C$$

where $G(y)$ is the antiderivative of $\frac{1}{g(y)}$ and $F(x)$ is the antiderivative of $f(x)$. This procedure gives an implicit formula formula for a function y that satisfies the differential equation. If we are given an initial condition, then we can solve for the integration constant C.

2.2 Example Problems

2.2.1 Initial Value Problems

Strategy: We first separate variables and integrate to recover a general form of the solution in terms of some yet to be determined integration constants. Next, we plug in the initial conditions to solve for the integration constants.

Problem 2.1. ($\star$) Suppose the velocity of a particle is given by

$$v(t) = \sin(t) - \cos(t).$$

Find the position function $s(t)$ of the particle given that $s(0) = 0$.

Solution 2.1. The rate of change of position is velocity, so we have the differential equation

$$v(t) = \frac{ds}{dt} = \sin(t) - \cos(t).$$

Separating variables and integrating, we have

$$\frac{ds}{dt} = \sin(t) - \cos(t) \Rightarrow \int ds = \int \sin(t) - \cos(t)\,dt \Rightarrow s = -\cos(t) - \sin(t) + C.$$

To solve for the integrating constant, since $s(0) = 0$, we have

$$0 = s(0) = -\cos(0) - \sin(0) + C = -1 + C \Rightarrow C = 1.$$

Therefore, the position is given by

$$s(t) = -\cos(t) - \sin(t) + 1.$$

Problem 2.2. ($\star\star$) Suppose the acceleration of a particle is given by

$$a(t) = t + 1.$$

Find the position function $s(t)$ of the particle given that $s(0) = 0$ and $s(1) = 2$.

Solution 2.2. The rate of change of velocity is acceleration, so we have the differential equation

$$\frac{dv}{dt} = t + 1.$$

Separating variables and integrating, we have

$$\frac{dv}{dt} = t + 1 \Rightarrow \int dv = \int t + 1\, dt \Rightarrow v = \frac{t^2}{2} + t + C.$$

The rate of change of position is velocity, so using the fact above, we have the differential equation

$$\frac{ds}{dt} = \frac{t^2}{2} + t + C.$$

Separating variables and integrating, we have

$$\frac{ds}{dt} = \frac{t^2}{2} + t + C \Rightarrow \int ds = \int \frac{t^2}{2} + t + C\, dt \Rightarrow s = \frac{t^3}{6} + \frac{t^2}{2} + Ct + D.$$

To solve for the integrating constants C and D, since $s(0) = 0$, we have

$$0 = s(0) = D \Rightarrow D = 0.$$

And since $s(1) = 2$, we have

$$2 = s(1) = \frac{1}{6} + \frac{1}{2} + C \Rightarrow C = 2 - \frac{1}{6} - \frac{1}{2} = \frac{4}{3}$$

Therefore, the position is given by

$$s(t) = \frac{t^3}{6} + \frac{t^2}{2} + \frac{4t}{3}.$$

2.2.2 Growth and Decay Problems

Problem 2.3. ($\star\star\star$) The rate of growth of a population P is modeled by

$$\frac{dP}{dt} = kP$$

where $k \neq 0$. Suppose that the initial population $P(0) = P_0$ for some constant $P_0 > 0$. Find the population function $P(t)$. How long will it take for the population to double?

Solution 2.3. Separating variables and integrating, we have

$$\frac{dP}{dt} = kP \Rightarrow \int \frac{dP}{P} = \int k dt \Rightarrow \ln|P| = kt + C$$

Since $P(t)$ must be positive in a reasonable model, $|P| = P$, so we can exponentiate both sides to conclude

$$P = e^{kt+C} = e^C e^{kt}.$$

To solve for the integrating constants C, since $P(0) = P_0$, we have

$$P_0 = P(0) = e^C e^{k\cdot 0} = e^C \Rightarrow \ln P_0 = C.$$

Therefore, the population is given by

$$P(t) = e^{\ln P_0} e^{kt} = P_0 e^{kt}.$$

To find the time for the population to double, we want to find the t such that $P(t) = 2P_0$. That is,

$$2P_0 = P_0 e^{kt} \Rightarrow e^{kt} = 2 \Rightarrow kt = \ln(2) \Rightarrow t = \frac{\ln(2)}{k}.$$

In particular, if $k < 0$ (we have a decreasing population) then we have our population will never double.

Problem 2.4. ($\star\star\star$) A pizza is put in a 200°C oven and heats up according to the differential equation

$$\frac{dH}{dt} = -k(H - 200), \text{ where } k > 0.$$

The pizza is put in the oven at 20°C and is removed 30 minutes later at a temperature of 120°C. Find the proportionality constant k.

Solution 2.4. The general solution to the differential equation is

$$\frac{dH}{dt} = -k(H - 200),$$

can be solved using separation of variables,

$$\frac{dH}{dt} = -k(H - 200) \Rightarrow \int \frac{dH}{(H - 200)} = \int -kt\, dt \Rightarrow \ln|H - 200| = -kt + C$$

Solving for H and using the fact $H - 200 < 0$ in a reasonable model, we see that

$$\ln|H - 200| = -kt + C \Rightarrow \ln(200 - H) = -kt + C \Rightarrow H = 200 - De^{-kt}$$

where $D = e^C > 0$. To find D, we can use the fact that at $H(0) = 20$,

$$20 = H(0) = 200 - D \implies D = 180.$$

Since $H(30) = 120$, we have

$$120 = H(30) = 200 - 180 \cdot e^{-30k} \implies k = -\frac{1}{30} \cdot \ln\frac{80}{180} \approx 0.027.$$

Problem 2.5. ($\star\star\star$) Find the general form of a function $y(x)$ such that

$$\frac{dy}{dx} = yx.$$

Solution 2.5. Separating variables and integrating, we have

$$\frac{dy}{dx} = yx \Rightarrow \int \frac{1}{y}\, dy = \int x\, dx \Rightarrow \ln|y| = \frac{x^2}{2} + A,$$

for some constant A. Solving for y we get

$$|y| = e^{\frac{x^2}{2} + A} \Rightarrow y = \pm e^A e^{\frac{x^2}{2}}.$$

If we define the non-zero constant $B = \pm e^A$, then we have

$$y = Be^{\frac{x^2}{2}}, \text{ where } B \text{ is some non-zero constant.}$$

However, notice that $y \equiv 0$ also satisfies $\frac{dy}{dx} = yx$, so $y = 0$ is also a solution. Therefore, the most general form of our solution is

$$y = Ce^{\frac{x^2}{2}}, \text{ where } C \text{ is some constant.}$$

Remark: We can check that $y(x) = Ce^{\frac{x^2}{2}}$ satisfies our differential equation. Notice that by the chain rule, we have

$$\frac{dy}{dx} = \frac{d}{dx} Ce^{\frac{x^2}{2}} = \underbrace{Ce^{\frac{x^2}{2}}}_{y} \cdot x = yx.$$

Remark: This example also explains how to remove the absolute value sign that appears when we take the antiderivative of $\frac{1}{y}$ and the usual approach one can take to absorb the resulting plus or minus sign into the constant of integration. For example, the solution for the population growth model in Problem 2.3 can be extended to negative populations or 0 initial populations using this argument.

3 Definite Integrals

Recall that the *definite integral* of $f(x)$ is defined as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* \in [a+(i-1)\Delta x, a+i\Delta x]$.

3.1 Basic Properties

Changing the Index:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

Linearity: If $c, d \in \mathbb{R}$

$$\int_a^b cf(x) + dg(x)\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx.$$

Monotonicity: If $f(x) \le g(x)$ for all $x \in [a,b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Splitting the Region of Integration: For any a, b, c,

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad \text{and} \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Symmetry: If f is integrable,

$$f \text{ odd} \implies \int_{-a}^a f(x)\,dx = 0 \quad \text{and} \quad f \text{ even} \implies \int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

3.1.1 Fundamental Theorem of Calculus

The fundamental theorem of calculus links the notions of differentiation and integration. Consider the area function of f from a to x,

$$A(x) = \int_a^x f(t)\,dt.$$

We expect that the rate of change of the area at the point $x = b$ should be proportional to the height of the function $f(b)$, that is $A'(b) = f(b)$. We can think of differentiation as the 'inverse of integration'. This notion is made precise with the fundamental theorem of calculus.

Theorem 1 (Fundamental Theorem of Calculus)**.** *Suppose f is a continuous function on $[a,b]$. Then*

1. *$A(x)$ is an antiderivative of f on (a,b). In particular, for all $x \in (a,b)$,*

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

2. *If F is an antiderivative of f on $[a,b]$, then*

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a).$$

Proof. We start by proving the first part of the fundamental theorem of calculus.

Proof of Part 1: Let $A(x)$ be the area function of f from a to x

$$A(x) = \int_a^x f(t)\,dt.$$

We can compute the derivative using the limit definition. Let $x \in (a,b)$, we have

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}.$$

Let $m(h) = \min_{t\in[x-|h|,x+|h|]} f(t)$ and $M(h) = \max_{t\in[x-|h|,x+|h|]} f(t)$ be the corresponding maximum and minimum of $f(t)$ on the interval $[x-|h|, x+|h|]$. These quantities exist by the Extreme Value Theorem because f is continuous and the interval is closed. We can now apply the squeeze theorem to compute our limit. For $t \in [x-|h|, x+|h|]$,

$$\begin{aligned} m(h) \le f(t) \le M(h) &\Rightarrow \frac{\int_x^{x+h} m(h)\,dt}{h} \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le \frac{\int_x^{x+h} M(h)\,dt}{h} \qquad \text{also holds for } h<0 \\ &\Rightarrow \frac{m(h)(x+h-x)}{h} \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le \frac{M(h)(x+h-x)}{h} \\ &\Rightarrow m(h) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le M(h). \end{aligned}$$

Notice that by continuity, we have $\lim_{h\to 0} m(h) = f(x)$ and $\lim_{h\to 0} M(h) = f(x)$ so

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} = f(x).$$

Proof of Part 2: From the Part 1, we know that $A(x)$ is an antiderivative of $f(x)$ on $[a,b]$. If $F(x)$ is another antiderivative of f, then we know that $A(x) = F(x) + C$ for some integration constant C and all $x \in [a,b]$. To find the integration constant, we evaluate our function at $x = a$ and conclude

$$A(a) = F(a) + C \implies 0 = F(a) + C \implies C = -F(a).$$

Therefore,

$$\int_a^b f(t)\,dt = A(b) = F(b) + C = F(b) - F(a).$$

□

Remark. The fundamental theorem of calculus says we can think of differentiation and integration as "inverse" operations. For example, the first part of the fundamental theorem states that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

so differentiation "cancels" the integration. On the other hand, the second part of the fundamental theorem of calculus states that

$$\int_a^x \frac{d}{dt} f(t)\,dt = f(x) - f(a),$$

so integration "cancels" differentiation (at least up to a constant).

3.2 Example Problems

3.2.1 Properties of Definite Integrals

Problem 3.1. ($\star$) If $\int_1^6 f(x)\,dx = 4$, $\int_1^2 f(x)\,dx = 3$, and $\int_5^6 f(x)\,dx = 7$, find $\int_2^5 f(x)\,dx$.

Solution 3.1. Using the fact that

$$\int_1^6 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx + \int_5^6 f(x)\,dx,$$

we have

$$\int_2^5 f(x)\,dx = \int_1^6 f(x)\,dx - \int_1^2 f(x)\,dx - \int_5^6 f(x)\,dx = 4 - 3 - 7 = -6.$$

Problem 3.2. ($\star\star$) The cumulative density function of the exponential random variable is a function defined on $[0, \infty)$ given by

$$F_\lambda(x) = \lambda \int_0^x e^{-\lambda t}\,dt,$$

where $\lambda > 0$ is a fixed constant. Express the definite integral $\int_5^9 e^{-\lambda t}\,dt$ in terms of $F_\lambda(x)$.

Solution 3.2. We start by splitting the region of integration

$$\int_5^9 e^{-\lambda t}\,dt = \int_5^0 e^{-\lambda t}\,dt + \int_0^9 e^{-\lambda t}\,dt = -\int_0^5 e^{-\lambda t}\,dt + \int_0^9 e^{-\lambda t}\,dt.$$

We now multiply and divide by λ to conclude

$$\int_5^9 e^{-\lambda t}\,dt = \frac{1}{\lambda}\left(\lambda \int_0^9 e^{-\lambda t}\,dt - \lambda \int_0^5 e^{-\lambda t}\,dt\right) = \frac{1}{\lambda} \cdot (F_\lambda(9) - F_\lambda(5)).$$

Problem 3.3. ($\star\star$) Suppose that $f(x)$ is an odd function, i.e. $f(-x) = -f(x)$. Prove that its integral $F(x) = \int_a^x f(t)\,dt$ is even for all choices of $a \in \mathbb{R}$.

Solution 3.3. It suffices to show that $F(-x) = F(x)$, that is $F(x) - F(-x) = 0$. This follows immediately from the properties of integration,

$$F(x) - F(-x) = \int_a^x f(t)\,dt - \int_a^{-x} f(t)\,dt = \int_a^x f(t)\,dt + \int_{-x}^a f(t)\,dt = \int_{-x}^x f(t)\,dt = 0,$$

since $f(t)$ is odd, so its integral around a symmetric interval is 0 by symmetry.

3.2.2 Application of the Fundamental Theorem of Calculus Part 1

Problem 3.4. ($\star$) Let $g(x) = \int_{-5}^x e^{-t^2}\,dt$. Find $g'(x)$.

Solution 3.4. By the fundamental theorem of calculus, we have

$$g'(x) = e^{-x^2}.$$

Problem 3.5. ($\star\star$) Let $g(x) = \int_0^{\ln(x)} e^{-t}\,dt$. Find $g'(x)$.

Solution 3.5. Let $h(x) = \int_0^x e^{-t}\,dt$. Notice that $g(x) = h(\ln(x))$. Therefore, by the fundamental theorem of calculus and the chain rule, we have

$$g'(x) = h'(\ln(x)) \cdot \frac{d}{dx}\ln(x) = e^{-\ln(x)} \cdot \frac{1}{x} = e^{\ln(x^{-1})} \cdot \frac{1}{x} = \frac{1}{x^2}.$$

We used the fact that $h'(x) = \frac{d}{dx}\int_0^x e^{-t}\,dt = e^{-x}$ by the fundamental theorem of calculus.

Remark: If we used the second part of the fundamental theorem of calculus, then

$$g(x) = \int_0^{\ln(x)} e^{-t}\,dt = -e^{-t}\Big|_{t=0}^{t=\ln(x)} = -e^{-\ln(x)} + 1 = e^{\ln(x^{-1})} + 1 = -\frac{1}{x} + 1.$$

Differentiating this, we see

$$g'(x) = \frac{d}{dx}\Big(-\frac{1}{x} + 1\Big) = \frac{1}{x^2}.$$

Problem 3.6. ($\star\star$) Let $g(x) = \int_{-x^2}^{4x} \sin^2(t)\,dt$. Find $g'(x)$.

Solution 3.6. We start by splitting the region of integration

$$g(x) = \int_{-x^2}^{4x} \sin^2(t)\,dt = \int_{-x^2}^{0} \sin(t)\,dt + \int_0^{4x} \sin^2(t)\,dt = -\int_0^{-x^2} \sin^2(t)\,dt + \int_0^{4x} \sin^2(t)\,dt.$$

By the fundamental theorem of calculus and the chain rule, we have

$$\begin{aligned} g'(x) &= -\frac{d}{dx}\int_0^{-x^2} \sin^2(t)\,dt + \frac{d}{dx}\int_0^{4x} \sin^2(t)\,dt && \text{linearity} \\ &= -\sin^2(-x^2) \cdot \frac{d}{dx}(-x^2) + \sin^2(4x) \cdot \frac{d}{dx}4x && \text{fundamental theorem \& chain rule} \\ &= 2x\sin^2(-x^2) + 4\sin^2(4x). \end{aligned}$$

The reasoning for the chain rule is the same as the previous problem.

3.2.3 Application of the Fundamental Theorem of Calculus Part 2

Problem 3.7. ($\star$) Find the area under the curve of e^{-2x} on the interval $[0, 9]$.

Solution 3.7. If suffices to compute the following definite integral

$$\int_0^9 e^{-2x}\,dx.$$

It is easy to check that $F(x) = -\frac{1}{2}e^{-2x}$ is an antiderivative of e^{-2x}. Therefore, by the fundamental theorem of calculus,

$$\int_0^9 e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_{x=0}^{x=9} = -\frac{1}{2}e^{-2\cdot 9} + \frac{1}{2}e^{-2\cdot 0} = \frac{1}{2} - \frac{1}{2}e^{-18}.$$

Remark: We can actually compute the area explicitly using Riemann sums. Recall that for $r \neq 0$, the sum of a geometric series is given by

$$\sum_{i=0}^{n-1} r^i = \Big(\frac{1-r^n}{1-r}\Big).$$

Using the left endpoint Riemann sum, we have

$$\lim_{n\to\infty} \frac{9}{n} \cdot \sum_{i=1}^{n} e^{-\frac{18(i-1)}{n}} \overset{j=i-1}{=} \lim_{n\to\infty} \frac{9}{n} \cdot \sum_{j=0}^{n-1} \big(e^{-\frac{18}{n}}\big)^j = \lim_{n\to\infty} \frac{9}{n} \cdot \Big(\frac{1-e^{-18}}{1-e^{-\frac{18}{n}}}\Big) = \frac{1}{2} - \frac{1}{2}e^{-18}$$

using L'Hôpital's rule.

Problem 3.8. ($\star\star$) Compute the limit

$$\lim_{n\to\infty} \frac{3}{n}\Big(\sqrt{1+\frac{3}{n}} + \sqrt{1+2\cdot\frac{3}{n}} + \sqrt{1+3\cdot\frac{3}{n}} + \cdots + \sqrt{1+n\cdot\frac{3}{n}}\Big).$$

Solution 3.8. The summation appears to be a right Riemann sum. Recall that the right Riemann sum is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\big(a + i\cdot\Delta x\big)\cdot\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\Big(a + i\cdot\frac{b-a}{n}\Big)\cdot\frac{b-a}{n}.$$

If we choose $a = 1$, $b = 4$ and $f(x) = \sqrt{x}$, then by the definition of the definite integral, we have

$$\int_1^4 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+i\Delta x}\cdot\Delta x = \lim_{n\to\infty} \frac{3}{n}\Big(\sqrt{1+\frac{3}{n}} + \sqrt{1+2\cdot\frac{3}{n}} + \cdots + \sqrt{1+n\cdot\frac{3}{n}}\Big).$$

Therefore, using the fundamental theorem of calculus, the infinite sum is equal to

$$\int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_{x=1}^{x=4} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$$

Remark. Notice that if we were to choose $a = 0$, $b = 3$ and $f(x) = \sqrt{1+x}$, then we also have

$$\int_0^3 \sqrt{1+x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+i\Delta x}\cdot\Delta x = \lim_{n\to\infty} \frac{3}{n}\Big(\sqrt{1+\frac{3}{n}} + \sqrt{1+2\cdot\frac{3}{n}} + \cdots + \sqrt{1+n\cdot\frac{3}{n}}\Big).$$

Therefore, using the Fundamental theorem of calculus to compute this infinite sum, we have

$$\int_0^3 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_{x=0}^{x=3} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$$

Once we learn integration by substitution, we will see why these two integrals are equal.

Remark. Yet another choice is picking $a = 0$ and $b = 1$. In this case, $x_i^* = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$, so

$$\lim_{n\to\infty} \frac{3}{n}\Big(\sqrt{1+\frac{3}{n}} + \sqrt{1+2\cdot\frac{3}{n}} + \cdots + \sqrt{1+n\cdot\frac{3}{n}}\Big) = \sum_{i=1}^{n} 3\sqrt{1+3\cdot\frac{i}{n}}\cdot\frac{1}{n} = \sum_{i=1}^{n} 3\sqrt{1+3x_i^*}\cdot\Delta x.$$

Therefore, using the Fundamental theorem of calculus to compute this integral, we have

$$\sum_{i=1}^{n} 3\sqrt{1+3x_i^*}\cdot\Delta x = \int_0^1 3\sqrt{1+3x}\,dx = \frac{2}{3}(1+3x)^{3/2}\Big|_{x=0}^{x=1} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$$