1 Applications of the Chain Rule

We go over several examples of applications of the chain rule to compute derivatives of more complicated functions.

Chain Rule: If $z = f(y)$ and $y = g(x)$ then

$$\frac{d}{dx}(f \circ g)(x) = \Big(\frac{d}{dx} f \circ g\Big)(x) \cdot \frac{d}{dx} g(x) = f'(g(x)) \cdot g'(x) \quad \text{or equivalently} \quad \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

The chain rule is used as the main tool to solve the following classes for problems:

1. *Implicit Differentiation*: The chain rule can be used to compute derivatives of implicit functions
$$F(x, y(x)) = 0$$
where F is a function of two variables x and y.

2. *Logarithmic Differentiation*: By first taking the logarithm of both sides, we can compute derivatives of
$$y(x) = f(x)^{g(x)}.$$

3. *Inverse Functions Differentiation*: The chain rule is used to derive the derivative of the inverse function formula
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{(\frac{d}{dx} f \circ f^{-1})(x)} = \frac{1}{f'(f^{-1}(x))}.$$

4. *Related Rates*: There are word problems where both y and x depend on some related variable t. The goal is to compute the rate of change of $y(x)$ with respect to t.

1.1 Example Problems

1.1.1 Implicit Differentiation

Strategy: If y cannot be written explicitly as a function of x, then we can still compute the derivatives.

1. We differentiate both sides of the equation with respect to x and multiply a term by $\frac{dy}{dx}$ whenever the derivative 'hits' the y term.
2. After computing the derivative, we solve for $\frac{dy}{dx}$ and leave our answer in terms of x and y.
3. We can evaluate the derivative at the point $(x_0, y(x_0))$ by plugging in the point $x = x_0$ and $y = y(x_0)$ into the derivative.

Problem 1. ($\star$) Consider the implicit function

$$-2x + 2e^y = x^2 + y^2 + xy + 3.$$

Find the $\frac{dy}{dx}$ when $x = -1$ and $y = 0$.

Solution 1. We differentiate both sides with respect to x,

$$\begin{aligned}
&\frac{d}{dx}(-2x + 2e^y) = \frac{d}{dx}(x^2 + y^2 + xy + 3) \\
&\Rightarrow -2 + 2e^y \frac{dy}{dx} = 2x + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} \qquad \text{Product Rule \& Chain Rule} \\
&\Rightarrow 2e^y \frac{dy}{dx} - 2y\frac{dy}{dx} - x\frac{dy}{dx} = 2x + y + 2 \\
&\Rightarrow \frac{dy}{dx} = \frac{2x + y + 2}{2e^y - 2y - x}.
\end{aligned}$$

Plugging in the point $x = -1$ and $y = 0$ into the formula above, we have

$$\frac{dy}{dx}\Big|_{x=-1,y=0} = \frac{2x + y + 2}{2e^y - 2y - x}\Big|_{x=-1,y=0} = 0.$$

Problem 2. ($\star\star$) Consider the implicit function

$$\sin y + \cos x = 1.$$

Find $\frac{d^2y}{dx^2}$ using implicit differentiation.

Solution 2. We differentiate both sides with respect to x,

$$\frac{d}{dx}\big(\sin y + \cos x\big) = \frac{d}{dx}1 \Rightarrow \cos(y)\frac{dy}{dx} - \sin(x) = 0 \Rightarrow \frac{dy}{dx} = \sin(x)\sec(y).$$

Differentiating this again, we have

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\sin(x)\sec(y) \Rightarrow \frac{d^2y}{dx^2} &= \cos(x)\sec(y) + \sec(y)\tan(y)\sin(x)\frac{dy}{dx} && \text{Product Rule \& Chain Rule} \\ \Rightarrow \frac{d^2y}{dx^2} &= \cos(x)\sec(y) + \sin^2(x)\sec^2(y)\tan(y). && \frac{dy}{dx} = \sin(x)\sec(y) \end{aligned}$$

1.1.2 Logarithmic Differentiation

Strategy: We want to compute the derivatives of functions of the form

$$y(x) = f(x)^{g(x)}.$$

By taking the logarithm of both sides, we have

$$\ln(y) = g(x)\ln(f(x)).$$

This function can be differentiated implicitly using the same strategy as the last section. Taking the logarithm of both sides of our equation can also be used to solve complicated quotient rule problems.

Remark: Logarithmic differentiation also works if $y(x) < 0$ for some values of x. To justify this, we can take the absolute value of both sides, followed by the natural log of both sides, and use the fact that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

This can be proved using the fact that the derivative of an even function is odd, and using an odd extension of $\frac{d}{dx}\ln x = \frac{1}{x}$ to $x < 0$.

Problem 1. ($\star$) Compute the derivative of

$$f(x) = x^x.$$

Solution 1. We set $y = f(x)$ and take the logarithm of both sides,

$$y = x^x \Rightarrow \ln(y) = x\ln(x).$$

Implicitly differentiating both sides, we have

$$\begin{aligned} \ln(y) = x\ln(x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} &= \ln(x) + 1 && \text{Product Rule and Chain Rule} \\ \Rightarrow \frac{dy}{dx} &= y(\ln(x) + 1) \\ \Rightarrow \frac{dy}{dx} &= x^x(\ln(x) + 1). && y = x^x \end{aligned}$$

Problem 2. ($\star\star$) Let

$$f(x) = \frac{2e^{\sqrt{x^2+1}}\sqrt{x+4}(x^2+2x+2)}{(x+1)^5}.$$

Find $f'(0)$.

Solution 2. Suppose that $x \geq 0$ and set $y = f(x)$. We start by taking the logarithm of both sides,

$$y = \frac{2e^{\sqrt{x^2+1}}\sqrt{x+4}(x^2+2x+2)}{(x+1)^5} \Rightarrow \ln(y) = \ln(2)+\sqrt{x^2+1}+\frac{1}{2}\ln(x+4)+\ln(x^2+2x+2)-5\ln(x+1).$$

Implicitly differentiating both sides, we have

$$\begin{aligned}
&\ln(y) = \ln(2) + \sqrt{x^2+1} + \frac{1}{2}\ln(x+4) + \ln(x^2+2x+2) - 5\ln(x+1) \\
&\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x + \frac{1}{2}\frac{1}{x+4} + \frac{2x+2}{x^2+2x+2} - 5\frac{1}{x+1} \\
&\Rightarrow \frac{dy}{dx} = y\cdot\Big(\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x + \frac{1}{2}\frac{1}{x+4} + \frac{2x+2}{x^2+2x+2} - 5\frac{1}{x+1}\Big).
\end{aligned}$$

When $x = 0$, we have $y = f(0) = \frac{2e^{\sqrt{x^2+1}}\sqrt{x+4}(x^2+2x+2)}{(x+1)^5}\Big|_{x=0} = 8e$ we have

$$f'(0) = \frac{dy}{dx}\Big|_{x=0,y=8e} = 8e\Big(\frac{1}{8}+1-5\Big) = -31e.$$

Remark: This problem can also be solved using the quotient rule. The computation is more cumbersome if we use the quotient rule.

Problem 3. ($\star\star\star$) Compute the derivative of

$$f(x) = x^{(x^x)}.$$

Solution 3. We set $y = f(x)$ and take the logarithm of both sides,

$$y = x^{(x^x)} \Rightarrow \ln(y) = x^x\ln(x).$$

This derivative is still hard to compute explicitly, so we take the logarithm of both sides again,

$$\ln(y) = x^x\ln(x) \Rightarrow \ln(\ln(y)) = \ln(x^x\ln(x)) = x\ln x + \ln(\ln(x)).$$

Implicitly differentiating both sides, we have

$$\begin{aligned}
\ln(\ln(y)) = x\ln x + \ln(\ln(x)) &\Rightarrow \frac{1}{\ln(y)}\cdot\frac{1}{y}\cdot\frac{dy}{dx} = \ln(x)+1+\frac{1}{\ln(x)}\cdot\frac{1}{x} && \text{Product Rule and Chain Rule} \\
&\Rightarrow \frac{dy}{dx} = y\ln(y)\Big(\ln(x)+1+\frac{1}{x\ln(x)}\Big) \\
&\Rightarrow \frac{dy}{dx} = x^{(x^x)}(x^x\ln(x))\Big(\ln(x)+1+\frac{1}{x\ln(x)}\Big) && y = x^{(x^x)},\ \ln(y) = x^x\ln(x) \\
&\Rightarrow \frac{dy}{dx} = x^{x^x+1}\Big(\ln^2(x)+\ln(x)+\frac{1}{x}\Big).
\end{aligned}$$

Problem 4. ($\star\star\star$) Prove the quotient rule

$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}.$$

Solution 4. We will use logarithmic differentiation. Set $y = \frac{f(x)}{g(x)}$. Since y may be less than 0, we first take the absolute value of both sides followed by the logarithm,

$$\ln|y(x)| = \ln|f(x)| - \ln|g(x)|.$$

Implicitly differentiating both sides, we have

$$\begin{aligned}
\ln|y| = \ln|f(x)| - \ln|g(x)| &\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{f(x)}f'(x) - \frac{1}{g(x)}g'(x) && \text{Chain Rule}\\
&\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{f(x)g(x)}\\
&\Rightarrow \frac{dy}{dx} = y\cdot\Big(\frac{g(x)f'(x) - f(x)g'(x)}{f(x)g(x)}\Big)\\
&\Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. && y = \frac{f(x)}{g(x)}
\end{aligned}$$

Note: The computations above work under the assumption that $y(x) \neq 0$.

1.1.3 Inverse Functions Differentiation

Problem 1. ($\star\star$) Prove the formula for the derivative of the inverse function

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{(\frac{d}{dx}f\circ f^{-1})(x)}.$$

Solution 1. By the cancellation laws, we have

$$(f\circ f^{-1})(x) = x.$$

Differentiating both sides and using the chain rule, we have

$$\frac{d}{dx}(f\circ f^{-1})(x) = \frac{d}{dx}x \Rightarrow \Big(\frac{d}{dx}f\circ f^{-1}\Big)(x)\cdot\frac{d}{dx}f^{-1}(x) = 1 \Rightarrow \frac{d}{dx}f^{-1}(x) = \frac{1}{(\frac{d}{dx}f\circ f^{-1})(x)}.$$

Problem 2. ($\star\star$) Let $f(x) = x + \sin(x)$. Find $(f^{-1})'(0)$.

Solution 2. Notice that $x+\sin(x)$ is one-to-one on $\mathbb{R}$, but its inverse is impossible to express in terms of functions we have encountered so far. However, we can still find the derivative of the inverse using the formula for the derivative of the inverse function.

Notice that $f(0) = 0 + \sin(0) = 0$, so we have $f^{-1}(0) = 0$. Since $f'(x) = 1 + \cos(x)$, the formula for the inverse derivative implies

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{1+\cos(0)} = \frac{1}{2}.$$

Problem 3. ($\star\star$) Using the formula for the derivative of the inverse function, show that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Solution 3. We will use the formula

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{(\frac{d}{dx}f \circ f^{-1})(x)}$$

with $f(x) = \sin(x)$. Since $f^{-1}(x) = \sin^{-1}(x)$ and $\frac{d}{dx}\sin(x) = \cos(x)$, the formula implies

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))}.$$

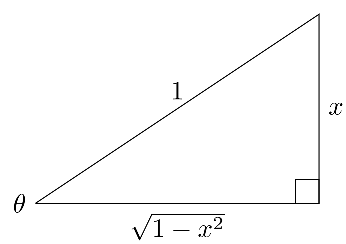
We now want to simplify the function $\cos(\sin^{-1}(x))$ without using trigonometric identities. This type of problem was introduced in Week 1:

Geometric Solution: We first find the domain of our function. We have $D_{\sin^{-1}(x)} = [-1, 1]$ and $D_{\cos(x)} = \mathbb{R}$ and $D_{1/x} = \{x \neq 0\}$, so our domain consists of points in $D_{\sin^{-1}(x)}$ such that

$$\cos(\sin^{-1}(x)) \neq 0 \Rightarrow \sin^{-1}(x) \neq \frac{\pi}{2}, -\frac{\pi}{2} \Rightarrow x \neq \sin\Big(\frac{\pi}{2}\Big), \sin\Big(-\frac{\pi}{2}\Big) \Rightarrow x \neq \pm 1.$$

Therefore, the domain of our function is $(-1, 1)$.

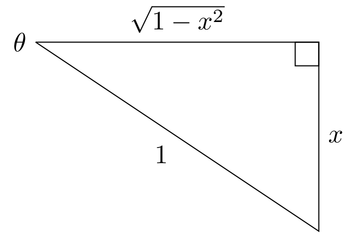
Case $x \geq 0$: We first consider the case such that $x > 0$ on our domain. On this region, we have $\theta = \sin^{-1}(x) \in [0, \frac{\pi}{2}]$ (the first quadrant). The triangle corresponding to $\sin(\theta) = x$ in the first quadrant is given by



From this triangle, we see

$$\frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}} \text{ for } x \in [0, 1).$$

Case $x < 0$: We first consider the case such that $x > 0$ on our domain. On this region, we have $\theta = \sin^{-1}(x) \in [-\frac{\pi}{2}, 0]$ (the fourth quadrant). The triangle corresponding to $\sin(\theta) = x$ in the fourth quadrant is given by



Notice that $x < 0$, so the side opposite the θ is positive. From this triangle, we see

$$\frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}} \text{ for } x \in (-1, 0).$$

Combining the two domains, we have

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \text{ for } x \in (-1, 1).$$

Problem 4. ($\star\star$) Using the formula for the derivative of the inverse function, show that

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}.$$

Solution 4. We will use the formula

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{(\frac{d}{dx}f \circ f^{-1})(x)}$$

with $f(x) = \sec(x)$. Since $f^{-1}(x) = \sec^{-1}(x)$ and $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, the formula implies

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))}.$$

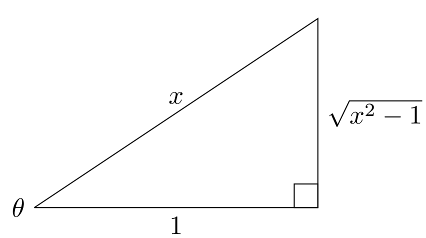
We now want to simplify the function $\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))$ without using trigonometric identities. This type of problem was introduced in Week 1:

Geometric Solution: We first find the domain of our function. We have $D_{\sec^{-1}(x)} = (-\infty, 1] \cup [1, \infty)$, $D_{\tan(x)} = \{x \neq \frac{2k+1}{2}\pi\}$, and $D_{1/x} = \{x \neq 0\}$. Notice that the range of $\sec^{-1}(x) = [0, \pi, 2) \cup (\pi/2, \pi]$ so $\tan(\sec^{-1}(x))$ is defined for all $x \in D_{\sec^{-1}(x)}$. Since $D_{1/x} = \{x \neq 0\}$ and the domain of $\sec^{-1}(x)$ does not include 0, our function is defined for points in $D_{\sec^{-1}(x)}$ such that

$$\tan(\sec^{-1}(x)) \neq 0 \Rightarrow \sec^{-1}(x) \neq n\pi \Rightarrow \sec^{-1}(x) \neq 0, \pi \Rightarrow x \neq \sec(0), \sec(\pi) \Rightarrow x \neq -1, 1.$$

Therefore, the domain is of the function is $(-\infty, -1) \cup (1, \infty)$.

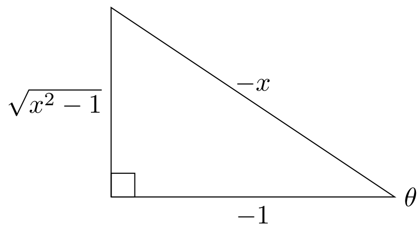
Case $x > 0$: We first consider the case such that $x > 0$ on our domain. On this region, we have $\theta = \sec^{-1}(x) \in [0, \frac{\pi}{2}]$ (the first quadrant). The triangle corresponding to $\sec(\theta) = x$ in the first quadrant is given by



From this triangle, we see

$$\frac{1}{\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))} = \frac{1}{\sec(\theta)\tan(\theta)} = \frac{1}{x\sqrt{x^2-1}} \text{ for } x \in (1, \infty).$$

Case $x < 0$: We now consider the case such that $x < 0$ on our domain. On this region, we have $\theta = \sec^{-1}(x) \in [\frac{\pi}{2}, \pi]$ (the second quadrant). The triangle corresponding to $\sec(\theta) = x$ in the second quadrant is given by



From this triangle, we see

$$\frac{1}{\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))} = \frac{1}{\sec(\theta)\tan(\theta)} = \frac{-1}{x\sqrt{x^2-1}} \text{ for } x \in (-\infty, -1).$$

Combining the two domains, we have

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \text{ for } x \in (-\infty,-1) \cup (1,\infty).$$

1.1.4 Related Rates

Strategy: The goal is to find a function (or an implicit function) that connects the quantities we have information about the rate of change. We then use the chain rule and differentiate both sides to find the corresponding rates of change.

Problem 1. ($\star\star$) A cylindrical tank with radius 5m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

Solution 1. We first summarize the information in the problem:

Given: Let V be the volume of the water in the tank. Since the tank is cylindrical with radius 5, we know that $V = \pi 5^2 h$, where h is the height of the water level. We also know that $\frac{dV}{dt} = 3$.

Goal: We want to find $\frac{dh}{dt}$.

Finding the Equation: We now use the information given to solve the problem. We first find a formula that relates V and h. Using the volume formula, we have

$$V = 25\pi h.$$

Solving for the Required Rate: To find $\frac{dh}{dt}$, we take the derivative of both sides with respect to t,

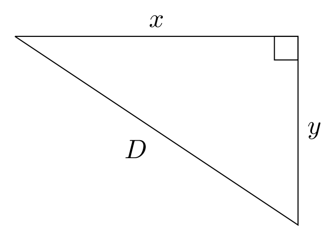
$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{25\pi}\frac{dV}{dt} \Rightarrow \frac{dh}{dt} = \frac{3}{25\pi},$$

since $\frac{dV}{dt} = 3$. Therefore, the height of the water is increasing at $\frac{3}{25\pi}$ m/min.

Problem 2. ($\star\star$) Two cars start moving from the same point. One travels south at 60 km/h and the other travels west at 25 km/h. At what rate is the distance between the cars increasing two hours later?

Solution 2. We first summarize the information in the problem:

Given: Let x be the position of the car traveling west, and let y be the position of the car traveling south, and let D be the distance the distance between the cars,



We also know that $\frac{dx}{dt} = 25$ and $\frac{dy}{dt} = 60$.

Goal: We want to find $\frac{dD}{dt}$ when $t = 2$.

Finding the Equation: We now use the information given to solve the problem. We first find a formula that relates D with x and y. Using the pythagorean theorem, we have

$$D^2 = x^2 + y^2.$$

Solving for the Required Rate: To find $\frac{dD}{dt}$, we take the derivative of both sides with respect to t,

$$\frac{d}{dt}D^2 = \frac{d}{dt}(x^2 + y^2) \Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dD}{dt} = D^{-1}\Big(x\frac{dx}{dt} + y\frac{dy}{dt}\Big).$$

We want to compute this quantity when $t = 2$. When $t = 2$, we have $x = 2 \cdot 25 = 50$ $y = 2 \cdot 60 = 120$ and $D = \sqrt{x^2 + y^2} = 130$. Furthermore, since $\frac{dx}{dt} = 25$ and $\frac{dy}{dt} = 60$, we have

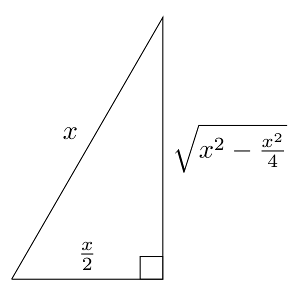
$$\frac{dD}{dt} = \frac{1}{130}\Big(50 \cdot 25 + 120 \cdot 60\Big) = 65.$$

Therefore, the distance between the cars is increasing at 65km/h.

Problem 3. ($\star\star$) The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long.

Solution 3. We first summarize the information in the problem:

Given: Let x be the length of a side of the triangle. Half of the triangle is given by the triangle



We also know that $\frac{dx}{dt} = 10$.

Goal: Let A be the area of this triangle. We want to find $\frac{dA}{dt}$ when $x = 30$.

Finding the Equation: We now use the information given to solve the problem. We first find a formula that relates A with x. Since the area of the a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$, since $x > 0$

$$\frac{1}{2}A = \frac{1}{2} \cdot \frac{x}{2} \cdot \sqrt{x^2 - \frac{x^2}{4}} \Rightarrow A = \frac{x^2}{2} \cdot \frac{\sqrt{3}}{2}.$$

Solving for the Required Rate: To find $\frac{dA}{dt}$, we take the derivative of both sides with respect to t,

$$\frac{dA}{dt} = \frac{d}{dt}\frac{\sqrt{3}x^2}{4} \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}x}{2} \cdot \frac{dx}{dt} \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3} \cdot 30}{2} \cdot 10 = 150\sqrt{3},$$

since $x = 30$ and $\frac{dx}{dt} = 10$. Therefore, the area of the triangle is increasing at $150\sqrt{3}$ cm^2/min.