1 Differentiation

Recall that the derivative (or first derivative) of $y = f(x)$ is given by

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Some other common notation for the derivative of $y = f(x)$ include

$$f'(x), \quad \frac{d}{dx} f(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}.$$

The second derivative is the derivative of the first derivative,

$$\frac{d^2}{dx^2} f(x) = \frac{d}{dx}\Big(\frac{d}{dx} f(x)\Big).$$

Some other common notation for the second derivative of $y = f(x)$ include

$$f''(x), \quad \frac{d^2}{dx^2} f(x), \quad \frac{d^2 y}{dx^2}, \quad \frac{d^2 f}{dx^2}.$$

1.1 Table of Derivatives

Elementary Functions

$$\frac{d}{dx} x^a = a x^{a-1}, \quad a \in \mathbb{R} \tag{1}$$

$$\frac{d}{dx} |x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{2}$$

Exponential and Logarithms

$$\frac{d}{dx} e^x = e^x \tag{3}$$

$$\frac{d}{dx} \ln(x) = \frac{1}{x} \tag{4}$$

$$\frac{d}{dx} a^x = \ln(a) \cdot a^x \tag{5}$$

$$\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)} \tag{6}$$

Trigonometric Functions

$$\frac{d}{dx} \sin(x) = \cos(x) \tag{7}$$

$$\frac{d}{dx} \cos(x) = -\sin(x) \tag{8}$$

$$\frac{d}{dx} \tan(x) = \sec^2(x) \tag{9}$$

$$\frac{d}{dx} \cot(x) = -\csc^2(x) \tag{10}$$

$$\frac{d}{dx} \sec(x) = \tan(x) \sec(x) \tag{11}$$

$$\frac{d}{dx} \csc(x) = -\cot(x) \csc(x) \tag{12}$$

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}} \tag{13}$$

$$\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}} \tag{14}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2} \tag{15}$$

$$\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1 + x^2} \tag{16}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2 - 1}} \tag{17}$$

$$\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2 - 1}} \tag{18}$$

Hyperbolic Functions

$$\frac{d}{dx} \sinh(x) = \cosh(x) \tag{19}$$

$$\frac{d}{dx} \cosh(x) = \sinh(x) \tag{20}$$

$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x) \tag{21}$$

1.2 Basic Properties

We now state several rules that will allow us to differentiate combinations of the basic functions

Addition:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Scaling: If $c \in \mathbb{R}$,

$$\frac{d}{dx}cf(x) = c \cdot \frac{d}{dx}f(x).$$

Product Rule:

$$\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = \frac{d}{dx}f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx}g(x).$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}.$$

Chain Rule: If $z = f(y)$ and $y = g(x)$ then

$$\frac{d}{dx}(f \circ g)(x) = \Big(\frac{d}{dx}f \circ g\Big)(x) \cdot \frac{d}{dx}g(x) = f'(g(x)) \cdot g'(x) \quad \text{or equivalently} \quad \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

Derivative of Inverse: If $f^{-1}(x)$ exists, then

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{(\frac{d}{dx}f \circ f^{-1})(x)} = \frac{1}{f'(f^{-1}(x))}.$$

1.3 Tangent and Normal Line Approximations

Using the derivative, we can construct the tangent lines and normal lines to the curve $y = f(x)$. If $f'(a)$ exists, then

1. Tangent Line: The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a)$$

2. Normal Line: The normal line to the curve $y = f(x)$ at the point $(a, f(a))$ is given by

$$y = f(a) - \frac{1}{f'(a)}(x - a).$$

1.4 Rates of Change

The derivative tells us the instantaneous rate of change of a function with respect to one of its input.

For example, let $s(t)$ denote the position of the particle at time t. The derivative of s tells us the instantaneous rate of change of the position with respect to time. That is, the velocity (instantaneous velocity) $v(t)$ of the particle is given by the first derivative of the position $v(t) = s'(t)$.

Similarly, the derivative of v tells us the instantaneous rate of change of the velocity with respect to time. That is, the acceleration of the particle, is the first derivative of the velocity (or equivalently, the second derivative of the position) $a(t) = v'(t) = s''(t)$.

1.5 Example Problems

1.5.1 Using the Limit Definition to Derive Limits

Problem 1. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = x^7$.

Solution 1. We use the alternate definition of the derivative,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

In this case, we have

$$\begin{aligned} f'(x) &= \lim_{y \to x} \frac{y^7 - x^7}{y - x} \\ &= \lim_{y \to x} \frac{(y-x)(y^6 + y^5x + y^4x^2 + y^3x^3 + y^2x^4 + yx^5 + x^6)}{y - x} && \text{Difference of Powers} \\ &= \lim_{y \to x} (y^6 + y^5x + y^4x^2 + y^3x^3 + y^2x^4 + yx^5 + x^6) \\ &= 7x^6. \end{aligned}$$

Problem 2. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = \sqrt{x}$.

Solution 2. From the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - x}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Problem 3. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = \ln(x)$ for $x > 0$.

Solution 3. From the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) \\ &= \ln \lim_{h \to 0} \left(1 + \frac{1}{x} \cdot h\right)^{\frac{1}{h}} && \text{Continuity} \\ &= \ln(e^{1/x}) && \lim_{h \to 0} \left(1 + ah\right)^{\frac{1}{h}} = \lim_{n \to \pm\infty} \left(1 + \frac{a}{n}\right)^{\frac{1}{n}} = e^a \\ &= \frac{1}{x}. \end{aligned}$$

Problem 4. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = \sin(x)$.

Solution 4. From the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \sin(x)\cdot \lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} && \text{Limit Laws} \\ &= \cos(x). && \lim_{h\to 0}\frac{\sin(h)}{h} = 1, \quad \lim_{h\to 0}\frac{\cos(x)-1}{x} = 0 \end{aligned}$$

1.5.2 Using the Basic Properties to Evaluate Derivatives

Problem 1. ($\star$) Using the formula for the derivative of the inverse, find the derivative of e^x.

Solution 1. We will use the formula

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{(\frac{d}{dx} f \circ f^{-1})(x)}$$

with $f(x) = \ln(x)$. Since $f^{-1}(x) = e^x$, and $\frac{d}{dx}\ln(x) = \frac{1}{x}$, the formula implies

$$\frac{d}{dx} e^x = \frac{1}{\frac{1}{e^x}} = e^x.$$

Problem 2. ($\star\star$) Given the curve $y = (4x-5)^3(x^2+5x-5)^7$, find the tangent and normal line to the curve when $x = 1$.

Solution 2. We have

$$\begin{aligned} &\frac{d}{dx}(4x-5)^3(x^2+5x-5)^7 \\ &= \frac{d}{dx}(4x-5)^3 \cdot (x^2+5x-5)^7 + (4x-5)^3 \cdot \frac{d}{dx}(x^2+5x-5)^7 && \text{product rule} \\ &= 3(4x-5)^2 \cdot 4 \cdot (x^2+5x-5)^7 + (4x-5)^3 \cdot 7 \cdot (x^2+5x-5)^6(2x+5). && \text{chain rule} \end{aligned}$$

Therefore,

$$f'(1) = 3\cdot 4\cdot (4-5)^2 \cdot (1^2+5-5)^7 + (4-5)^3 \cdot 7(1^2+5-5)^6(2+5) = 12 - 49 = -37.$$

Since $f(1) = -1$, using the formula we have that

$$y_{tangent} = f'(1)(x-1) + f(1) = -37(x-1) - 1 = -37x + 36$$

and

$$y_{normal} = -\frac{1}{f'(1)}(x-1) + f(1) = \frac{1}{37}(x-1) - 1 = \frac{1}{37}x - \frac{38}{37}.$$

Problem 3. ($\star$) Compute the derivative of

$$f(x) = e^{\frac{x}{x-1}}.$$

Solution 3. We have

$$\begin{aligned}\frac{d}{dx}e^{\frac{x}{x-1}} &= e^{\frac{x}{x-1}} \cdot \frac{d}{dx}\frac{x}{x-1} && \text{chain rule}\\ &= e^{\frac{x}{x-1}} \cdot \frac{(x-1)-x}{(x-1)^2} && \text{quotient rule}\\ &= \frac{-e^{\frac{x}{x-1}}}{(x-1)^2}.\end{aligned}$$

Problem 4. ($\star$) Compute the derivative of

$$f(x) = \frac{\cos(x)}{1+\tan(x)}.$$

Solution 4. We have

$$\begin{aligned}\frac{d}{dx}\frac{\cos(x)}{1+\tan(x)} &= \frac{-(1+\tan(x))\sin(x) - \sec^2(x)\cos(x)}{(1+\tan(x))^2} && \text{quotient rule}\\ &= -\frac{(1+\tan(x))\sin(x)+\sec(x)}{(1+\tan(x))^2}.\end{aligned}$$

Problem 5. ($\star\star$) Compute the second derivative of

$$f(x) = x\sec(x).$$

Solution 5. We have

$$\begin{aligned}\frac{d}{dx}x\sec(x) &= \sec(x) + x\sec(x)\tan(x) && \text{product rule}\\ &= \sec(x)(1+x\tan(x)).\end{aligned}$$

Taking the derivative again, we have

$$\begin{aligned}\frac{d^2}{dx^2}x\sec(x) &= \frac{d}{dx}\sec(x)(1+x\tan(x))\\ &= \sec(x)\tan(x)(1+x\tan(x)) + \sec(x)(\tan(x)+x\sec^2(x)) && \text{product rule}\\ &= x\sec(x)\tan^2(x) + 2\sec(x)\tan(x) + x\sec^3(x).\end{aligned}$$