1 Continuity

Definition 1. A function is *continuous at a* if

$$\lim_{x \to a} f(x) = f(a).$$

Similarly, a function is *continuous from the right at a*, if

$$\lim_{x \to a^+} f(x) = f(a),$$

and *continuous from the left at a*, if

$$\lim_{x \to a^-} f(x) = f(a).$$

In the definitions above, we implicitly assumed that the all the quantities are well defined, that is, the appropriate limits $\lim_{x \to a} f(x)$ or $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ exists, and $f(a)$ exists.

Definition 2. A function is *continuous at on an interval* if is continuous at every point in the interval (with the appropriate one-sided notion of continuity at an endpoint). For example, a function f is continuous on (a, b) if f is continuous at all $x \in (a, b)$. A function f is continuous on $(a, b]$ if f is continuous at all $x \in (a, b)$ and f is continuous from the left at the endpoint b.

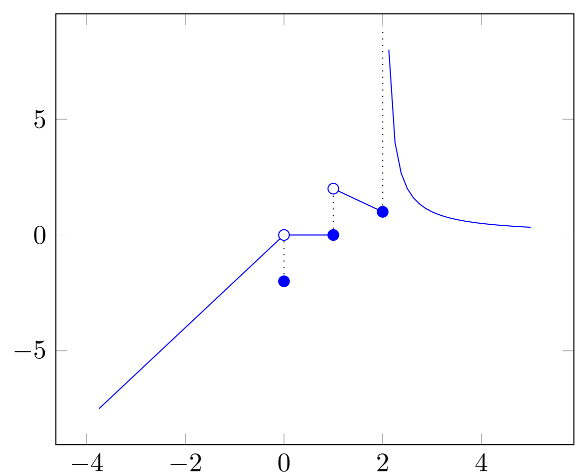
All the basic functions introduced in Week 1 are continuous at all points where the function is defined. Furthermore, all basic function operations $f + g$, $f \cdot g$, $f \circ g$, etc, preserve continuity:

1. Addition: If f and g are continuous at a, then $f + g$ is continuous at a.
2. Multiplication: If f and g are continuous at a, then $f \cdot g$ is continuous at a.
3. Composition: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

This means that basic combinations of functions in Week 1 are also continuous where defined.

Definition 3. There are 3 main types of discontinuities

1. Removable Discontinuity: A *removable discontinuity* occurs at a when $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$ or $f(a)$ is not defined.
2. Jump Discontinuity: A *jump discontinuity* occurs at a when both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ different and finite. Note that we do not need $f(a)$ to exist.
3. Infinite Discontinuity: An *infinite discontinuity* occurs at a when one or both of the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ are infinite. Note that we do not need $f(a)$ to exist.



Example 1. This graph has a removable discontinuity at $x = 0$, a jump discontinuity at $x = 1$, and an infinite discontinuity at $x = 2$.

1.1 Three Theorems about Continuous Functions

We state several important theorems related to continuous functions.

1.1.1 Intermediate Value Theorem:

Theorem 1 (Intermediate Value Theorem). *If f is continuous on the interval $[a, b]$ and N is an intermediate value between $f(a)$ and $f(b)$, that is*

$$\min(f(a), f(b)) < N < \max(f(a), f(b)),$$

then there exists a $c \in (a, b)$ such that $f(c) = N$.

The Intermediate Value Theorem provides a nice way to solve inequalities with continuous functions.

Corollary 1. *If f is continuous on (a, b) and $f(x) \neq 0$ for all $x \in (a, b)$, then $f(x) > 0$ for all $x \in (a, b)$ or $f(x) < 0$ for all $x \in (a, b)$.*

1.1.2 Differentiable implies Continuous:

Definition 4. Recall that the derivative of $y = f(x)$ is given by

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

We say a function is *differentiable at a* if $f'(a)$ exists, and *differentiable on an interval* (a, b) if $f'(x)$ exists for all $x \in (a, b)$.

Every differentiable function is continuous:

Theorem 2. *If f is differentiable at a, then f is continuous at a.*

However, *not* every continuous function is differentiable, so the converse the statement is false.

Example 2. Since $\lim_{x\to 0} |x| = 0 = |0|$, $|x|$ is continuous at 0. However, the function is not differentiable at 0 because the left and right derivatives give different values,

$$\lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \frac{h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \frac{-h}{h} = -1.$$

Therefore the $\lim_{h\to 0} \frac{|x+h|-|x|}{h}$ does not exist when $x = 0$, so the function is not differentiable at 0.

1.1.3 Continuous implies integrable

Definition 5. Recall that the definite integral of $f(x)$ is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$. A function is *integrable* if the limit on the right exists for all ways of choosing x_i^*.

Every continuous function is integrable:

Theorem 3. *If f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx$ exists.*

However, *not* every integrable function is continuous, so the converse of the statement is false.

Example 3. The function $f(x) = 0$ for $x \in [0, 1]$ and $f(x) = 1$ for $x \in [1, 2]$ is not continuous, but it integrable on $[0, 2]$. It is easy to check from the definition that $\int_0^2 f(x)\,dx = 1$. For any x_i^*, we have

$$1 - \frac{1}{n} \leq \sum_{i=1}^{n} f(x_i^*)\,\Delta x \leq 1 + \frac{1}{n} \implies \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = 1.$$

by the squeeze theorem.

1.2 Example Problems

1.2.1 Checking Continuity

Problem 1.1. ($\star$) Given that

$$f(x) = \begin{cases} \ln(x-1)+b & x > 2 \\ x^2 & x = 2 \\ e^{x+a} & x < 2 \end{cases}$$

find the values of a and b such that f is continuous on $\mathbb{R}$.

Solution 1.1. We know that the continuity is preserved under basic function operations, so $f(x)$ is continuous where it is defined. In particular, this means that $f(x)$ is continuous on $(2,\infty)$ and $(-\infty, 2)$. We only have to check the about the potential discontinuity at $x = 2$. Notice that

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (\ln(x-1)+b) = \ln(1) + b = b$$

and by continuity,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} e^{x+a} = e^{\lim_{x\to 2^+} x+a} = e^{2+a}.$$

Since $f(2) = 2^2 = 4$, we need $b = e^{2+a} = 4$ for our function to be continuous. In particular, we need

$$b = 4 \text{ and } e^{2+a} = 4 \implies a = \ln(4) - 2 \text{ and } b = 4.$$

Problem 1.2. ($\star\star$) Let

$$f(x) = \frac{x^2-4}{x-a}.$$

1. Find the value(s) of a such that $f(x)$ has a removable discontinuity.
2. Find the value(s) of a such that $f(x)$ has a infinite discontinuity.
3. Find the value(s) of a such that $f(x)$ has a jump discontinuity.

Solution 1.2. Notice that

$$f(x) = \frac{x^2-4}{x-a} = \frac{(x-2)(x+2)}{x-a}.$$

We have that $f(x)$ is not defined at $x = a$, and our function is continuous everywhere else. If $a = 2$, notice that

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4,$$

so $\lim_{x\to 2} f(x)$ exists. Therefore, there is a removable discontinuity at $x = 2$. Similarly, if $a = -2$, notice that

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x\to -2}(x-2) = -4,$$

so $\lim_{x\to -2} f(x)$ exists. Therefore, there is a removable discontinuity at $x = -2$. For $a \neq \pm 2$, we have

$$\lim_{x\to a^+} \frac{(x-2)(x+2)}{x-a} = \frac{(a-2)(a+2)}{0^+} = \begin{cases} \infty & |a| > 2 \\ -\infty & |a| < 2 \end{cases}$$

since the numerator is not 0. Therefore, we have an infinite discontinuity at $x = a$.

To summarize, the function $f(x)$ has a removable discontinuity if $a = \pm 2$ and an infinite discontinuity if $a \neq \pm 2$.

1.2.2 Application of the Intermediate Value Theorem

Problem 1.3. ($\star\star$) Show that the function

$$f(x) = x^2 - 3 - \ln(x)$$

has at least 2 roots in the interval $(0, 2)$.

Solution 1.3. Notice that $f(x)$ is continuous on $(0, 2)$. To show a continuous function has a root in $(0, 2)$, it suffices to find two points a and b in $(0, 2)$ such that $f(a) > 0$ and $f(b) < 0$. To show that a function has at least 2 roots, we essentially have to do this procedure twice, and double check that our points lie in disjoint intervals.

One can check that $f(0.01) = 0.01^2 - 3 - \ln(0.01) \approx 1.6025 > 0$, and $f(1) = -2 < 0$, $f(1.99) = 1.99^2 - 3 - \ln(1.99) \approx 0.2720 > 0$. Therefore, since our function is continuous on $[0.01, 1]$ and $f(1) < 0 < f(0.01)$, there exists a root $c_1 \in (0.01, 1)$ such that $f(c_1) = 0$. Similarly, since our function is continuous on $[1, 1.99]$ and $f(1) < 0 < f(1.99)$, there exists a root $c_2 \in (1, 1.99)$ such that $f(c_2) = 0$. Since the intervals $(0.01, 1)$ and $(1, 1.99)$ are disjoint, we have $c_1 \neq c_2$, so $f(x)$ has at least 2 roots in the interval $(0, 2)$.

Problem 1.4. ($\star\star$) Find all x such that

$$f(x) = e^{2x} - 6e^x + 8 > 0.$$

Solution 1.4. Notice that $f(x)$ is continuous for all $x \in \mathbb{R}$. We can apply the Corrolary 1 to find the regions where $f(x) > 0$. We first find the roots $f(x) = 0$,

$$e^{2x} - 6e^x + 8 = (e^x - 4)(e^x - 2) = 0 \Leftrightarrow e^x - 2 = 0 \text{ or } e^x - 4 = 0 \Leftrightarrow x = \ln(2) \text{ or } x = \ln(4).$$

Therefore, $f(x)$ is non-zero on the intervals $(-\infty, \ln(2)), (\ln(2), \ln(4)), (\ln(4), \infty)$. By Corrolary 1, it suffices to check one point in each of the intervals to determine the sign of f for all x in the interval.

1. $(-\infty, \ln(2))$: Take $0 \in (-\infty, \ln(2))$, $f(0) = e^0 - 6e^0 + 8 > 0$, so $f(x) > 0$ for all $x \in (-\infty, \ln(2))$
2. $(\ln(2), \ln(4))$: Take $\ln(3) \in (\ln(2), \ln(4))$, $f(\ln(3)) = e^{\ln(3^2)} - 6e^{\ln(3)} + 8 = 9 - 18 + 8 < 0$, so $f(x) < 0$ for all $x \in (\ln(2), \ln(4))$
3. $(\ln(4), \infty)$: Take $\ln(10) \in (\ln(4), \infty)$, $f(\ln(10)) = e^{\ln(10^2)} - 6e^{\ln(10)} + 8 = 100 - 60 + 8 > 0$, so $f(x) > 0$ for all $x \in (\ln(4), \infty)$

Therefore, $f(x) > 0$ for $x \in (-\infty, \ln(2)) \cup (\ln(4), \infty)$.

1.2.3 Proofs of Continuity Theorems

Problem 1.5. ($\star\star\star$) Prove Corollary 1: If f is continuous on (a, b) and $f(x) \neq 0$ for all $x \in (a, b)$, then $f(x) > 0$ for all $x \in (a, b)$ or $f(x) < 0$ for all $x \in (a, b)$.

Solution 1.5. We do a proof by contradiction. Suppose that f is continuous on (a, b) and $f(x) \neq 0$ for all $x \in (a, b)$ and there exists two points c and d in (a, b) such that $f(c) < 0$ and $f(d) > 0$. The intermediate value theorem then implies that there exists a point y between c and d such that $f(y) = 0$ contradicting the fact $f(x)$ is non-zero on (a, b).

Problem 1.6. ($\star\star\star$) Prove Theorem 2: If f is differentiable at a, then f is continuous at a.

Solution 1.6. Suppose f is differentiable at a. We need to show that $\lim_{x\to a} f(x) = f(a)$. Since f is differentiable, we know

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If we define $h = x - a$, then we have

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

which is the *alternate definition* of a derivative. Using the alternate definition, we have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} (f(x) - f(a) + f(a)) \\
&= \lim_{x\to a} \Big(\frac{f(x) - f(a)}{x - a}(x - a) + f(a)) \Big) \\
&= \Big(\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a} (x - a) + \lim_{x\to a} f(a)) \Big) && \text{Limit Laws} \\
&= f'(a) \cdot \lim_{x\to a} (x - a) + f(a) && \text{Alternate Definition} \\
&= f(a),
\end{aligned}$$

so f is continuous.

1.2.4 Limits of the composition of discontinuous functions

If f is discontinuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x))$ may not behave nicely. These examples are to illustrate the potential dangers when taking limits with a discontinuous function on the outside. In these cases, evaluating the function at a may not give the right answer even if we do not get an indeterminate form.

Problem 1.7. ($\star\star$) Consider the discontinuous function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and let $g(x) = x^2$. Compute $\lim_{x\to 0} f(g(x))$.

Solution 1.7. We first compute the composition of the functions,

$$f(g(x)) = f(x^2) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

since $x^2 = 0$ if and only if $x = 0$. Therefore, we have

$$\lim_{x\to 0} f(g(x)) = 0,$$

since $f(g(x))$ is equal to 0 for all $x \neq 0$.

Remark. We cannot take the limit inside function f in this problem. In this case, if we incorrectly used the property $\lim_{x\to 0} f(g(x)) = f(\lim_{x\to 0} g(x))$ then we will incorrectly conclude

$$\lim_{x\to 0} f(g(x)) = \lim_{x\to 0} f(x^2) = f(\lim_{x\to 0} x^2) = f(0) = 1.$$

The reason why we can't take the limit inside is that $f(x)$ is not continuous at 0, so the property does not hold. This is a reason why existence of the limits is not enough for the composition of functions.

Problem 1.8. ($\star\star$) Consider the discontinuous function

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and let $g(x) = x^2$. Compute $\lim_{x\to 0} f(g(x))$.

Solution 1.8. We first compute the composition of the functions,

$$f(g(x)) = f(x^2) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

since $x^2 > 0$ whenever $x \neq 0$ and $x^2 = 0$ when $x = 0$. Therefore, we have

$$\lim_{x\to 0} f(g(x)) = 1,$$

since $f(g(x))$ is equal to 1 for all $x \neq 0$.

Remark. In this example, even though the the limit $\lim_{x\to 0} f(x)$ does not exist (check that the left and right limits give different values), the limit of $f(g(x))$ still exists. If we tried to take the limit inside, $\lim_{x\to 0} f(g(x)) = f(\lim_{x\to 0} g(x))$ then we would have gotten the incorrect answer, for the same reasoning as the previous problem.

Problem 1.9. ($\star\star$) Consider the discontinuous function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Compute $\lim_{x\to 0} f(f(x))$.

Solution 1.9. We first compute the composition of the functions,

$$f(f(x)) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

Therefore, we have

$$\lim_{x\to 0} f(f(x)) = 1.$$

Remark. In this case, if we incorrectly used the property $\lim_{x\to 0} f(g(x)) = f(\lim_{x\to 0} g(x))$ then we get

$$\lim_{x\to 0} f(f(x)) = f(\lim_{x\to 0} f(x)) = f(0) = 1,$$

which is the correct answer. This just happened by chance, and the steps to arrive at this conclusion is completely wrong without more justification. This is also the usual counter example that disproves the claim that if $\lim_{g\to b} f(g) = L$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = \lim_{g\to b} f(g) = L$.