1 Continuity

Definition 1. A function is *continuous at a* if

$$\lim_{x\to a} f(x) = f(a).$$

Similarly, a function is *continuous from the right at a*, if

$$\lim_{x\to a^+} f(x) = f(a),$$

and *continuous from the left at a*, if

$$\lim_{x\to a^-} f(x) = f(a).$$

In the definitions above, we implicitly assumed that the all the quantities are well defined, that is, the appropriate limits $\lim_{x\to a} f(x)$ or $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ exists, and $f(a)$ exists.

Definition 2. A function is *continuous at on an interval* if is continuous at every point in the interval (with the appropriate one-sided notion of continuity at an endpoint). For example, a function f is continuous on (a, b) if f is continuous at all $x \in (a, b)$. A function f is continuous on $(a, b]$ if f is continuous at all $x \in (a, b)$ and f is continuous from the left at the endpoint b.

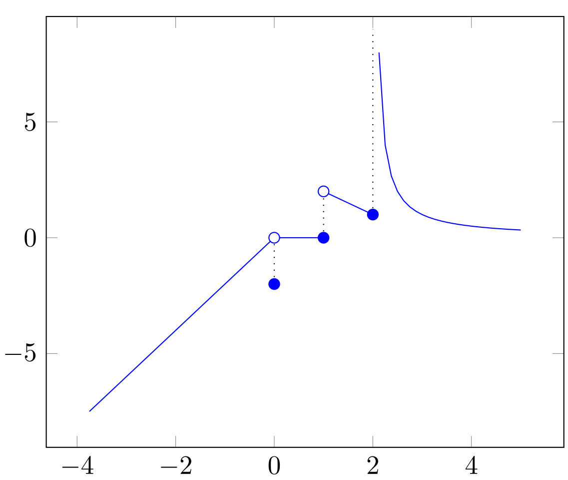
All the basic functions introduced in Week 1 are continuous at all points where the function is defined. Furthermore, all basic function operations $f + g$, $f \cdot g$, $f \circ g$, etc, preserve continuity:

1. Addition: If f and g are continuous at a, then $f + g$ is continuous at a.
2. Multiplication: If f and g are continuous at a, then $f \cdot g$ is continuous at a.
3. Composition: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

This means that basic combinations of functions in Week 1 are also continuous where defined.

Definition 3. There are 3 main types of discontinuities

1. Removable Discontinuity: A *removable discontinuity* occurs at a when $\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$ or $f(a)$ is not defined.
2. Jump Discontinuity: A *jump discontinuity* occurs at a when both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ different and finite. Note that we do not need $f(a)$ to exist.
3. Infinite Discontinuity: An *infinite discontinuity* occurs at a when one or both of the limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ are infinite. Note that we do not need $f(a)$ to exist.



Example 1: This graph has a removable discontinuity at $x = 0$, a jump discontinuity at $x = 1$, and an infinite discontinuity at $x = 2$.

1.1 Three Theorems about Continuous Functions

We state several important theorems related to continuous functions.

1.1.1 Intermediate Value Theorem:

Theorem 1 (Intermediate Value Theorem)**.** *If f is continuous on the interval $[a, b]$ and N is an intermediate value between $f(a)$ and $f(b)$, that is*

$$\min(f(a), f(b)) < N < \max(f(a), f(b)),$$

then there exists a $c \in (a, b)$ such that $f(c) = N$.

The Intermediate Value Theorem provides a nice way to solve inequalities involving continuous functions.

Corollary 1. *If f is continuous on (a, b) and $f(x) \neq 0$ for all $x \in (a, b)$, then $f(x) > 0$ for all $x \in (a, b)$ or $f(x) < 0$ for all $x \in (a, b)$*

Proof. We do a proof by contradiction. Suppose that f is continuous on (a, b) and $f(x) \neq 0$ for all $x \in (a, b)$ and there exists two points c and d in (a, b) such that $f(c) < 0$ and $f(d) > 0$. The intermediate value theorem then implies that there exists a point y between c and d such that $f(y) = 0$ contradicting the fact $f(x)$ is non-zero on (a, b). □

1.1.2 Differentiable implies Continuous:

We now state the relation between differentiable and continuous functions. Recall that the derivative of $y = f(x)$ is given by

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Definition 4. We say a function is *differentiable at a* if $f'(a)$ exists, and *differentiable on an interval* (a, b) if $f'(x)$ exists for all $x \in (a, b)$.

Every differentiable function is continuous, but *not* every continuous function is differentiable. This follows from the fact

Theorem 2. *If f is differentiable at a, then f is continuous at a.*

Proof. Suppose f is differentiable at a. We need to show that $\lim_{x\to a} f(x) = f(a)$. Since f is differentiable, we know

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If we define $h = x - a$, then we have

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

which is the *alternate definition* of a derivative. Using the alternate definition, we have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} (f(x) - f(a) + f(a)) \\
&= \lim_{x\to a} \Big(\frac{f(x) - f(a)}{x - a}(x - a) + f(a)) \Big) \\
&= \Big(\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) + \lim_{x\to a} f(a)) \Big) && \text{Limit Laws} \\
&= f'(a) \cdot \lim_{x\to a}(x - a) + f(a) && \text{Alternate Definition} \\
&= f(a),
\end{aligned}$$

so f is continuous. □

1.1.3 Continuous implies integrable

We now state the relationship between integrable and continuous functions. Recall that the definite integral of $f(x)$ is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* \in [a+(i-1)\Delta x, a+i\Delta x]$.

A function is called *integrable* provided the limit on the right exists. We have that every continuous function is integrable, but there are also many integrable functions that are not continuous.

Theorem 3. *If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx$ exists.*

The proof is out of the scope of this course, and usually requires a more general notion of a Riemann Sum.

1.2 Example Problems

1.2.1 Checking Continuity

Problem 1. ($\star$) Given that

$$f(x) = \begin{cases} \ln(x-1)+b & x>2 \\ x^2 & x=2 \\ e^{x+a} & x<2 \end{cases}$$

find the values of a and b such that f is continuous on $\mathbb{R}$.

Solution 1. We know that the continuity is preserved under basic function operations, so $f(x)$ is continuous where it is defined. In particular, this means that $f(x)$ is continuous on $(2,\infty)$ and $(-\infty,2)$. We only have to check the about the potential discontinuity at $x=2$. Notice that

$$\lim_{x\to 2^-}(\ln(x-1)+b) = \ln(1)+b = b$$

and by continuity,

$$\lim_{x\to 2^+} e^{x+a} = e^{\lim_{x\to 2^+} x+a} = e^{2+a}.$$

Since $f(2)=2^2=4$, we need $b=e^{2+a}=4$ for our function to be continuous. In particular, we need

$$b=4 \text{ and } e^{2+a}=4 \implies a=\ln(4)-2 \text{ and } b=4.$$

Problem 2. ($\star\star$) Let

$$f(x) = \frac{x^2-4}{x-a}.$$

1. Find the value(s) of a such that $f(x)$ has a removable discontinuity.
2. Find the value(s) of a such that $f(x)$ has a infinite discontinuity.
3. Find the value(s) of a such that $f(x)$ has a jump discontinuity.

Solution 2. Notice that

$$f(x) = \frac{x^2 - 4}{x - a} = \frac{(x-2)(x+2)}{x - a}.$$

We have that $f(x)$ is not defined at $x = a$, and our function is continuous everywhere else. If $a = 2$, notice that

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = 4,$$

so $\lim_{x\to 2} f(x)$ exists. Therefore, there is a removable discontinuity at $x = 2$. Similarly, if $a = -2$, notice that

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x \to -2} (x-2) = -4,$$

so $\lim_{x\to -2} f(x)$ exists. Therefore, there is a removable discontinuity at $x = -2$. For $a \neq \pm 2$, we have

$$\lim_{x \to a^+} \frac{(x-2)(x+2)}{x-a} = \frac{(a-2)(a+2)}{0^+} = \begin{cases} \infty & |a| > 2 \\ -\infty & |a| < 2 \end{cases}$$

since the numerator is not 0. Therefore, we have an infinite discontinuity at $x = a$.

To summarize, the function $f(x)$ has a removable discontinuity if $a = \pm 2$ and an infinite discontinuity if $a \neq \pm 2$.

1.2.2 Application of the Intermediate Value Theorem

Problem 1. ($\star\star$) Show that the function

$$f(x) = x^2 - 3 - \ln(x)$$

has at least 2 roots in the interval $(0, 2)$.

Solution 1. Notice that $f(x)$ is continuous on $(0, 2)$. To show a continuous function has a root in $(0, 2)$, it suffices to find two points a and b in $(0, 2)$ such that $f(a) > 0$ and $f(b) < 0$. To show that a function has at least 2 roots, we essentially have to do this procedure twice, and double check that our points lie in disjoint intervals.

One can check that $f(0.01) = 0.01^2 - 3 - \ln(0.01) \approx 1.6025 > 0$, and $f(1) = -2 < 0$, $f(1.99) = 1.99^2 - 3 - \ln(1.99) \approx 0.2720 > 0$. Therefore, since our function is continuous on $[0.01, 1]$ and $f(1) < 0 < f(0.01)$, there exists a root $c_1 \in (0.01, 1)$ such that $f(c_1) = 0$. Similarly, since our function is continuous on $[1, 1.99]$ and $f(1) < 0 < f(1.99)$, there exists a root $c_2 \in (1, 1.99)$ such that $f(c_2) = 0$. Since the intervals $(0.01, 1)$ and $(1, 1.99)$ are disjoint, we have $c_1 \neq c_2$, so $f(x)$ has at least 2 roots in the interval $(0, 2)$.

Problem 2. ($\star\star$) Find x such that

$$f(x) = e^{2x} - 6e^x + 8 > 0.$$

Solution 2. Notice that $f(x)$ is continuous for all $x \in \mathbb{R}$. We can apply the Corrolary 1 to find the regions where $f(x) > 0$. We first find the roots $f(x) = 0$,

$$e^{2x} - 6e^x + 8 = (e^x - 4)(e^x - 2) = 0 \Leftrightarrow e^x - 2 = 0 \text{ or } e^x - 4 = 0 \Leftrightarrow x = \ln(2) \text{ or } x = \ln(4).$$

Therefore, $f(x)$ is non-zero on the intervals $(-\infty, \ln(2)), (\ln(2), \ln(4)), (\ln(4), \infty)$. By Corrolary 1, it suffices to check one point in each of the intervals to determine the sign of f for all x in the interval.

1. $(-\infty, \ln(2))$: Take $0 \in (-\infty, \ln(2))$, $f(0) = e^0 - 6e^0 + 8 > 0$, so $f(x) > 0$ for all $x \in (-\infty, \ln(2))$
2. $(\ln(2), \ln(4))$: Take $\ln(3) \in (\ln(2), \ln(4))$, $f(\ln(3)) = e^{\ln(3^2)} - 6e^{\ln(3)} + 8 = 9 - 18 + 8 < 0$, so $f(x) < 0$ for all $x \in (\ln(2), \ln(4))$
3. $(\ln(4), \infty)$: Take $\ln(10) \in (\ln(4), \infty)$, $f(\ln(10)) = e^{\ln(10^2)} - 6e^{\ln(10)} + 8 = 100 - 60 + 8 > 0$, so $f(x) > 0$ for all $x \in (\ln(4), \infty)$

Therefore, $f(x) > 0$ for $x \in (-\infty, \ln(2)) \cup (\ln(4), \infty)$.