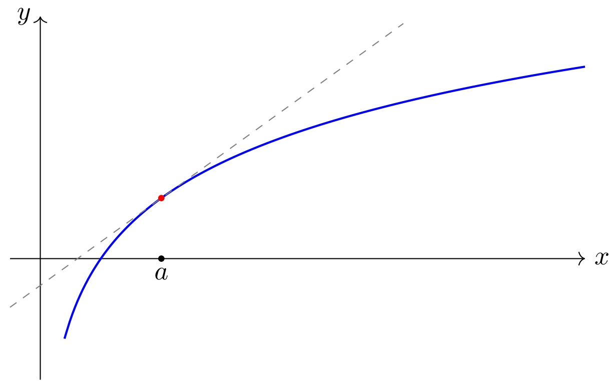
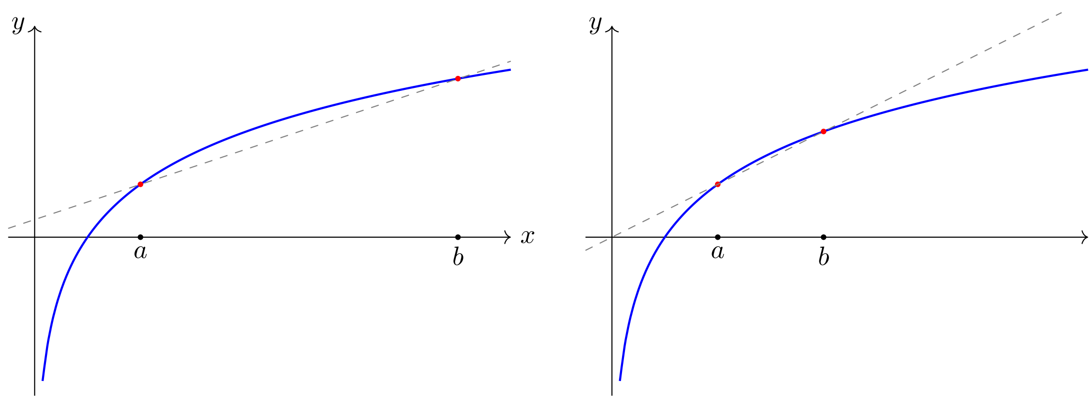
1 Tangent Line Problem

Question: *Given the graph of a function $y = f(x)$, what is the slope of the curve at the point $(a, f(a))$?*



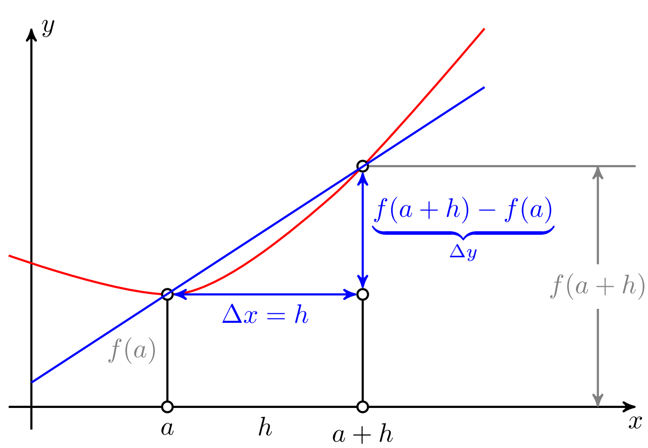
Our strategy is to approximate the slope by a limit of secant lines between points $(a, f(a))$ and $(b, f(b))$. The approximation improves as b gets closer and closer to a.



Definition 1. The slope of a secant line approximation for $y = f(x)$ between points $(a, f(a))$ and $(a + h, f(a + h))$ for $h > 0$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

The secant line approximation can be visualized below



Definition 2. The slope $f'(a)$ of the tangent line to $f(x)$ at point a is given by

$$f'(a) = \frac{dy}{dx}\Big|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This is the limit of secant of secant lines between the points $(a, f(a))$ and $(a+h, f(a+h))$ as $h \to 0$. If the number $f'(a)$ exists, then we say f is *differentiable* at a and we call the quantity $f'(a)$ the *derivative* of f at a.

1.1 Application to Velocity

Let $s(t)$ be the position of a particle at time t. In this context, Definition 1 and Definition 2 have the following interpretations

1. Secant Line: The *average velocity* v_{av} of the particle is given by the secant line approximation of the function $s(t)$ on the interval $a \leq t \leq b$,

$$v_{\text{av}} = \frac{s(b) - s(a)}{b - a}$$

2. Tangent Line: The *instantaneous velocity* v_{inst} is the tangent line of the function $s(t)$ at the point $x = a$

$$v_{\text{inst}} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

1.2 Example Problems

Useful Formulas: The equation of a tangent line approximation of the function f at the point $x = a$ is given by

$$\frac{y - f(a)}{x - a} = f'(a).$$

Problem 1. ($\star$) Let $f(x) = x^2 - 2$, find the secant line between the points $(1, f(1))$ and $(4, f(4))$

Solution 1. Taking $a = 1$ and $h = 3$ in our formula, we have

$$\frac{\Delta y}{\Delta x} = \frac{f(4) - f(1)}{4 - 1} = \frac{14 + 1}{3} = 5.$$

Problem 2. ($\star$) Suppose that the position of a particle moving horizontally on the x-axis is given by $s(t) = t^3 - 1$ for $t \in [0, 10]$.

a) Find the average velocity of the object on the time interval $[0, 5]$.

b) Find the instantaneous velocity at time $t = 1$.

Solution 2.

Part a) Taking $a = 0$ and $h = 5$ in our formula, the average velocity is given by

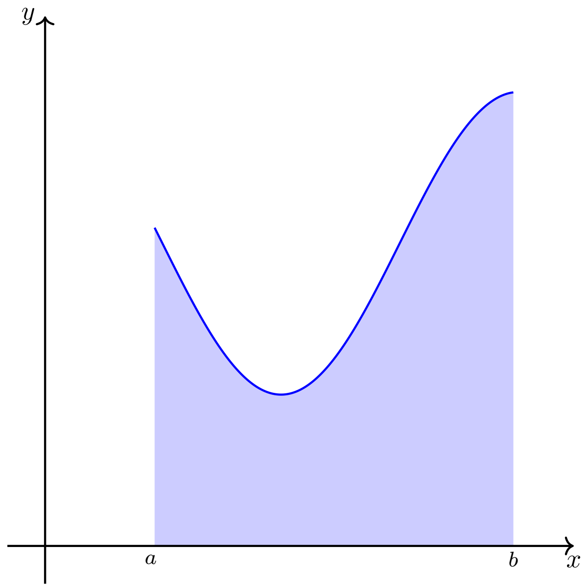
$$\frac{\Delta x}{\Delta t} = \frac{s(5) - s(0)}{5} = \frac{(5^3 - 1) - (-1)}{5} = 25.$$

Part b) The instantaneous velocity is given by

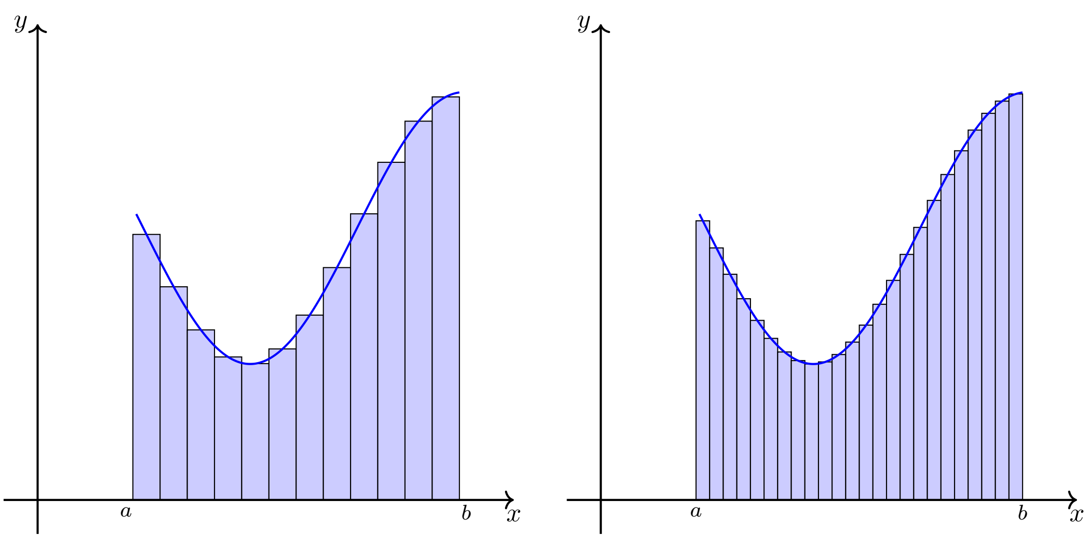
$$\frac{ds}{dt}\Big|_{t=1} = \lim_{h \to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h \to 0} \frac{h^3 + 3h^2 + 3h + 1 - 1}{h} = 3.$$

2 Area Problem

Question: *Given the graph of a function $y = f(x)$, what is the net area (the area above the x-axis and under the curve f minus the area below the x-axis and above the curve of f) of the graph between the points a and b?*



Our strategy is to divide the region $[a, b]$ into n subintervals and approximate the area by a limit of rectangles approximating our function. The approximation improves by taking n larger and larger.



Definition 3. The Riemann sum approximation of $\int_a^b f(x)\,dx$ on the interval $[a, b]$ with n uniform subintervals is given by

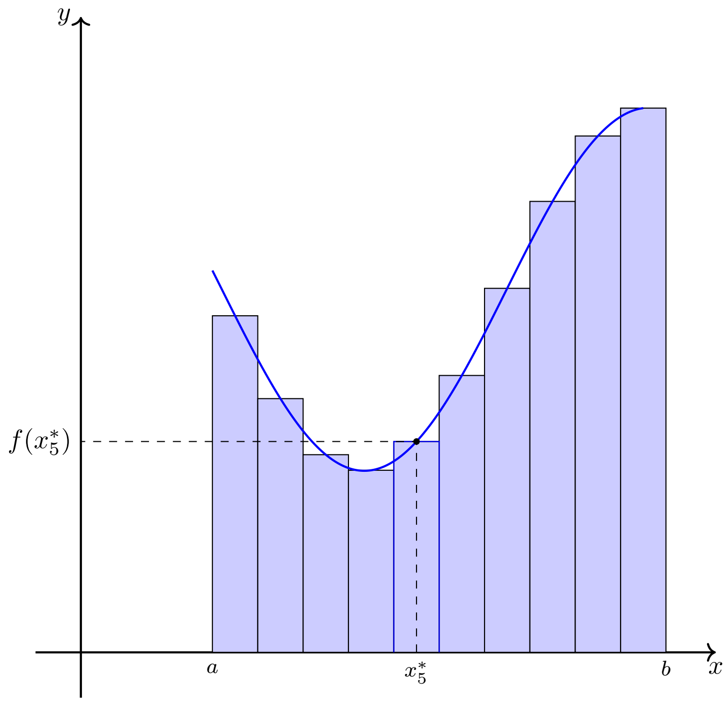
$$S_{[a,b]}(f) = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{(b-a)}{n}$ and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$. The approximate net area of the graph f is given by the Riemann Sum approximation.

Remark: We usually sample our function f at the right endpoint, midpoint, or left endpoint of each interval:

1. Right Riemann Sum: Take $x_i^* = a + i\Delta x$
2. Midpoint Riemann Sum: Take $x_i^* = a + (i - \frac{1}{2})\Delta x$
3. Left Riemann Sum: Take $x_i^* = a + (i - 1)\Delta x$

The midpoint approximation can be visualized below



Definition 4. The net area of the graph f on the interval $[a, b]$ is given by the definite integral of $f(x)$ on $[a, b]$. We call the quantity $\int_a^b f(x)\,dx$ the *definite integral* of f on $[a, b]$, and it is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{(b-a)}{n}$ and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$. This is the limit of Riemann sum approximations as $n \to \infty$. If the number $\int_a^b f(x)\,dx$ exists[1], then we say f is *integrable* on $[a, b]$.

2.1 Application to Velocity

Let $v(t)$ be the velocity of a particle at time t. In this context, Definition 4 has the following interpretations

1. Definite Integral of $|f|$: The *distance traveled* by the particle is given by the definite integral of $|v|$ on the interval $a \le t \le b$, which is given explicitly by the formula

$$\int_a^b |v(t)|\,dt.$$

[1]The limit has to exist and must all be identical for all choices of samples x_i^*.

2. Definite Integral of f: The *net distance traveled* (or *displacement*) d_v of the particle is given by the definite integral of $|v|$ on the interval $a \leq t \leq b$, which is given explicitly by the formula
$$\int_a^b v(t)\,dt.$$

2.2 Example Problems

Useful Formulas: The following formulas for the partial sums of a number will be useful to compute the Riemann Sums of certain functions

1. Sum of first n constants:
$$\sum_{i=1}^{n} 1 = n. \tag{1}$$
2. Sum of first n integers:
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{2}$$
3. Sum of first n squares:
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3}$$
4. Sum of first n cubes:
$$\sum_{i=1}^{n} i^3 = \Big(\frac{n(n+1)}{2}\Big)^2. \tag{4}$$

Problem 1. ($\star\star$) Approximate the area under the curve $y = f(x) = 2x$ above the x-axis on the interval $[0, 10]$ using 10 uniform subintervals and sampling $f(x)$ at the right endpoint of each interval.

Solution 1. We take $a = 0$, $b = 10$, and $n = 10$ in Definition 3. Since we are sampling at the right endpoints, we choose $x_i^* = i\Delta x \in [(i-1)\Delta x, i\Delta x]$ where $\Delta x = \frac{b-a}{n} = 1$. Therefore, using our formula, we have
$$\begin{aligned} S_{[0,10]}(f) &= \sum_{i=1}^{10} f(i\Delta x)\Delta x = \sum_{i=1}^{10} 2i = 2\sum_{i=1}^{10} i \\ &= 2\frac{10(10+1)}{2} = 110. \qquad \text{since } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \end{aligned}$$

Problem 2. ($\star\star\star$) Approximate the area under the curve $y = f(x) = 2x$ above the x-axis on the interval $[0, 10]$ using n uniform subintervals and sampling $f(x)$ at the right endpoint of each interval. What does the area converge to when we take $n \to \infty$.

Solution 2. We take $a = 0$, $b = 10$, with variable n in Definition 3. Since we are sampling at the right endpoints, we choose $x_i^* = i\Delta x \in [(i-1)\Delta x, i\Delta x]$ where $\Delta x = \frac{10-0}{n}$. Therefore, using our formula, we have
$$\begin{aligned} S_{[0,10]}(f) &= \sum_{i=1}^{n} f\Big(i\frac{10}{n}\Big)\Delta x = \sum_{i=1}^{n} 2\cdot\frac{10i}{n}\cdot\frac{10}{n} = \frac{200}{n^2}\sum_{i=1}^{n} i \\ &= \frac{200}{n^2}\cdot\frac{n(n+1)}{2} = 100\cdot\frac{n+1}{n}. \qquad \text{since } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \end{aligned}$$

Taking $n \to \infty$, we have

$$\lim_{n\to\infty} S_{[0,10]}(f) = \lim_{n\to\infty} 100 \cdot \frac{n+1}{n} = 100.$$

Note: The final answer is the same as

$$\int_0^{10} 2x\,dx = x^2\Big|_{x=0}^{x=10} = 100.$$

Problem 3. ($\star\star$) Approximate the area under the curve $y = f(x) = x^2$ above the x-axis on the interval $[0,1]$ using 100 uniform subintervals and sampling $f(x)$ at the left endpoint of each interval.

Solution 3. We take $a = 0$, $b = 1$, and $n = 100$ in Definition 3. Since we are sampling at the left endpoints, we choose $x_i^* = (i-1)\Delta x \in [(i-1)\Delta x, i\Delta x]$ where $\Delta x = \frac{1}{100}$. Therefore, using our formula, we have

$$\begin{aligned} S_{[0,1]}(f) = \sum_{i=1}^{100} f\Big((i-1)\Delta x\Big)\Delta x &= \sum_{i=1}^{100}\Big(\frac{i-1}{100}\Big)^2\frac{1}{100} \\ &= \frac{1}{100^3}\sum_{i=1}^{100}(i-1)^2 \\ &= \frac{1}{100^3}\sum_{i=0}^{99} j^2 && \text{by reindexing } j = i-1. \\ &= \frac{1}{100^3}\cdot\frac{99(100)(199)}{6} && \text{since } \sum_{j=0}^{n} j^2 = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \\ &= 0.32835. \end{aligned}$$

Problem 4. ($\star\star$) Approximate the area under the curve $y = f(x) = x^2$ above the x-axis on the interval $[1,5]$ using 100 uniform subintervals and sampling $f(x)$ at the right endpoint of each interval.

Solution 4. We take $a = 1$, $b = 5$, and $n = 100$ in Definition 3. Since we are sampling at the right endpoints, we choose $x_i^* = 1 + i\Delta x \in [1 + (i-1)\Delta x, 1 + i\Delta x]$ where $\Delta x = \frac{5-1}{100} = \frac{1}{25}$. Therefore, using our formula, we have

$$\begin{aligned} S_{[0,1]}(f) &= \sum_{i=1}^{100} f\Big(1 + i\Delta x\Big)\Delta x \\ &= \sum_{i=1}^{100}\Big(1 + \frac{i}{25}\Big)^2\cdot\frac{1}{25} \\ &= \frac{1}{25^3}\sum_{i=1}^{100}(25+i)^2 \\ &= \frac{1}{15625}\sum_{i=1}^{100}(625 + 50i + i^2) \\ &= \frac{1}{15625}\Big(625\sum_{i=1}^{100}1 + 50\sum_{i=1}^{100}i + \sum_{i=1}^{100}i^2\Big) \\ &= \frac{1}{15625}\Big(625\cdot 100 + 50\cdot\frac{100\cdot 101}{2} + \frac{100(101)(201)}{6}\Big) && \text{formulas (1), (2), (3)} \\ &= 41.8144. \end{aligned}$$

Problem 5. ($\star\star\star$) Approximate the area under the curve $y = f(x) = x^2$ above the x-axis on the interval $[0, 1]$ using n uniform subintervals and sampling $f(x)$ at the midpoint of each interval. What does the area converge to when we take $n \to \infty$.

Solution 5. We take $a = 0$, $b = 1$, with variable n in Definition 3. Since we are sampling at the midpoints of the intervals, we choose $x_i^* = (i - \frac{1}{2})\Delta x \in [(i-1)\Delta x, i\Delta x]$ where $\Delta x = \frac{1}{n}$. Therefore, using our formula, we have

$$\begin{aligned}
S_{[0,1]}(f) &= \sum_{i=1}^{n} f\Big(\Big(i - \frac{1}{2}\Big)\Delta x\Big)\Delta x \\
&= \sum_{i=1}^{n} \Big(\frac{2i-1}{2n}\Big)^2 \frac{1}{n} \\
&= \frac{1}{4n^3}\sum_{i=1}^{n}(2i-1)^2 \\
&= \frac{1}{4n^3}\sum_{i=1}^{n}(4i^2 - 4i + 1) \\
&= \frac{1}{4n^3}\Big(4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\Big) \\
&= \frac{1}{4n^3}\Big(4 \cdot \frac{n(n+1)(2n+1)}{6} - 4 \cdot \frac{n(n+1)}{2} + n\Big) \qquad \text{using formulas (1), (2), (3)} \\
&= \frac{n(n+1)(2n+1)}{6n^3} - \frac{(n+1)}{2n^2} + \frac{1}{4n^2}.
\end{aligned}$$

As $n \to \infty$, we have

$$\lim_{n\to\infty}\Big(\frac{n(n+1)(2n+1)}{6n^3} - \frac{(n+1)}{2n^2} + \frac{1}{4n^2}\Big) = \frac{1}{3}.$$

Remark: The final answer is the same as

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

Problem 6. ($\star$) Approximate the value of $\int_1^2 \ln(x)\,dx$ by using a left endpoint Riemann sum and 4 uniform subintervals.

Solution 6. We take $a = 1$, $b = 2$, and $n = 4$ in Definition 3. Since we are sampling at the left endpoints, we choose $x_i = 1 + (i-1)\Delta x \in [1 + (i-1)\Delta x, 1 + i\Delta x]$ where $\Delta x = \frac{b-a}{n} = \frac{1}{4}$. Therefore, using our formula, we have

$$\begin{aligned}
S_{[1,2]}(f) &= \sum_{i=1}^{4} f\Big(1 + \frac{i-1}{4}\Big)\frac{1}{4} \\
&= \frac{1}{4}\sum_{i=1}^{4}\ln\Big(1 + \frac{i-1}{4}\Big) \\
&= \frac{1}{4}\Big(\ln(1) + \ln(1.25) + \ln(1.5) + \ln(1.75)\Big) \approx 0.2970\ldots
\end{aligned}$$