1 Functions Defined in Terms of Integrals

Integrals allow us to define new functions in terms of the basic functions introduced in Week 1. Given a continuous function $f(x)$, consider the area function

$$F(x) = \int_a^x f(t)\,dt.$$

General Properties of $F(x)$: The following properties will allow us to sketch $F(x)$ even if the definite integral is impossible to simplify:

1. $F(x)$ is continuous where it is defined. (Fundamental theorem of calculus)
2. $F(a) = 0$. (Definition of the definite integral)
3. $F'(x) = f(x)$. (Fundamental theorem of calculus)
4. $F''(x) = f'(x)$. (Fundamental theorem of calculus)
5. If $a = 0$ and $f(x)$ is even, then $F(x)$ is odd. (Change of variables)
6. If $f(x)$ is odd, then $F(x)$ is even. (Change of variables)

1.1 The Natural Logarithm

Definition 1. For $x > 0$, the *natural logarithm* is defined by

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

Sketching the Curve: Using the basic properties of integral defined functions for $F(x) = \ln(x)$ we know that:

1. The x intercepts and derivatives of F are given by
$$F(1) = 0,\ F'(x) = \frac{1}{x},\ F''(x) = -\frac{1}{x^2}.$$
2. With some work, we can also show that $\lim_{x\to 0^+} F(x) = -\infty$ and $\lim_{x\to\infty} F(x) = \infty$.

Therefore, we can conclude that $F(x)$ is a strictly increasing concave down function that passes through the point $(1, 0)$. The second point also tells us there is a vertical asymptote at $x = 0$ and the integral diverges to ∞ as $x \to \infty$.

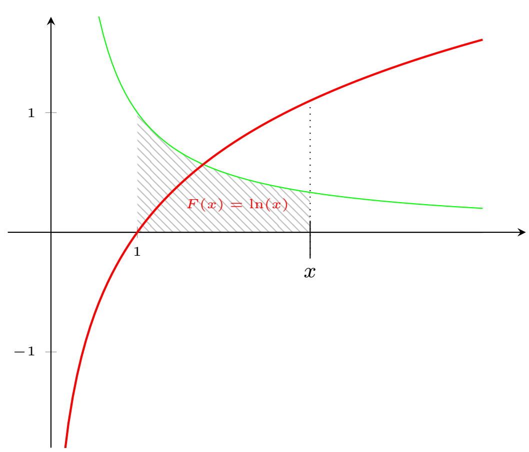


Figure 1: The graph of $f(x) = \frac{1}{x}$ and $F(x) = \ln(x)$ are displayed above. The value of $F(x)$ is the area under the curve of $\ln(x)$ between 1 and x.

1.2 The Error Function

Definition 2. For $x \in \mathbb{R}$, the *error function* is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Sketching the Curve: Using the basic properties of integral defined functions for $F(x) = \mathrm{erf}(x)$ we know that:

1. The x intercepts and derivatives of F are given by
$$F(0) = 0,\ F'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2},\ F''(x) = -\frac{4x}{\sqrt{\pi}} e^{-x^2}.$$
2. With some work, we can also show that $\lim_{x\to\infty} F(x) = 1$.
3. $F(x)$ is odd since $f(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$ is even and the lower limit of integration is 0.

Therefore, we can conclude for $x \geq 0$ that $F(x)$ is a strictly increasing concave down function that passes through the point $(0, 0)$. The second point also implies that $y = 1$ is a horizontal asymptote as $x \to \infty$. Since $F(x)$ is odd, we can recover the shape for $x < 0$ by reflecting around the origin.

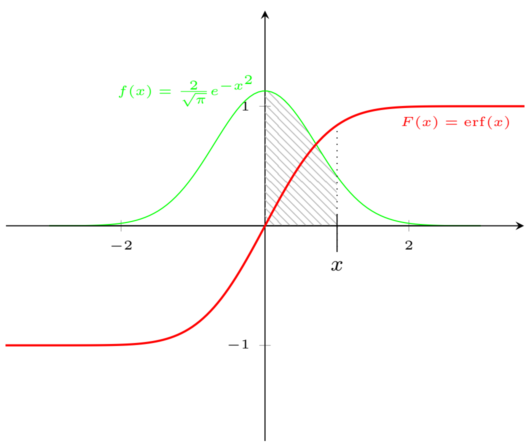


Figure 2: The graph of $f(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$ and $F(x) = \mathrm{erf}(x)$ are displayed above. The value of $F(x)$ is the area under the curve of $\frac{2}{\sqrt{\pi}} e^{-x^2}$ between 0 and x.

1.3 Examples of Other Integral Defined Functions

1. Fresnel Integrals:
$$S(x) = \int_0^x \sin(t^2)\, dt \quad \text{and} \quad C(x) = \int_0^x \cos(t^2)\, dt.$$
2. Sine Integral:
$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt.$$
3. Logarithmic Integral:
$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\, dt \quad \text{for} \quad x \geq 2.$$

1.4 Example Problems

1.4.1 Properties About Integral Defined Functions

Problem 1.1. ($\star$) Let

$$F(x) = \int_a^x f(t)\,dt.$$

Show that $F'(x) = f(x)$ and $F''(x) = f'(x)$.

Solution 1.1. By the first part of the fundamental theorem of calculus,

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Differentiating this again implies

$$F''(x) = f'(x).$$

Problem 1.2. ($\star\star$) Suppose $f(x)$ is even, that is $f(-x) = f(x)$. Show that the function

$$F(x) = \int_0^x f(t)\,dt$$

is an odd function.

Solution 1.2. It suffices to show $F(-x) = -F(x)$. Using the change of variables $u = -t$,

$$du = -dt, \qquad t = 0 \to u = 0, \qquad t = -x \to u = x$$

we have

$$\begin{aligned} F(-x) = \int_0^{-x} f(t)\,dt &= -\int_0^x f(-u)\,du \\ &= -\int_0^x f(u)\,du \qquad f(-u) = f(u) \\ &= -F(x). \end{aligned}$$

Problem 1.3. ($\star\star$) Suppose $f(x)$ is odd, that is $f(-x) = -f(x)$. Show that the function

$$F(x) = \int_a^x f(t)\,dt$$

is an even function.

Solution 1.3. It suffices to show that $F(-x) = F(x)$, that is $F(x) - F(-x) = 0$. This follows immediately from the properties of integration,

$$F(x) - F(-x) = \int_a^x f(t)\,dt - \int_a^{-x} f(t)\,dt = \int_a^x f(t)\,dt + \int_{-x}^a f(t)\,dt = \int_{-x}^x f(t)\,dt = 0,$$

since $f(t)$ is odd, so its integral around a symmetric interval is 0 by symmetry.

1.4.2 The Natural Logarithm

Problem 1.4. ($\star\star$) Using the integral definition of the natural logarithm, show that

$$\int \ln(x)\,dx = x\ln(x) - x + C.$$

Solution 1.4. We can integrate by parts to recover the formula for the antiderivative,

$\pm$	D	I
$+$	$\ln(x)$	1
$-\int$	$\frac{d}{dx}\ln(x)$	x

.

Since $\frac{d}{dx}\ln(x) = \frac{1}{x}$ by the fundamental theorem, we have

$$\int \ln(x)\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C.$$

Remark. It is easy to check that the $x\ln(x) - x + C$ is an antiderivative by simply differentiating.

Problem 1.5. ($\star\star\star$) Using the integral definition of the natural logarithm, show that

$$\ln(xy) = \ln(x) + \ln(y)$$

Solution 1.5. We want to write $\ln(xy)$ in terms of its integral definition. The trick is to "split" the integral

$$\begin{aligned}
\ln(xy) &= \int_1^{xy} \frac{1}{t}\,dt \\
&= \int_1^{x} \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt && \int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt \\
&= \int_1^{x} \frac{1}{t}\,dt + \int_1^{y} \frac{1}{u}\,du && u = \frac{t}{x},\ du = \frac{dt}{x},\ \int_x^{xy} dt \to \int_1^y du \\
&= \ln(x) + \ln(y).
\end{aligned}$$

1.4.3 The Error Function

Problem 1.6. ($\star\star$) Using the integral definition of the error function, show that

$$\int \operatorname{erf}(x)\,dx = x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}\cdot e^{-x^2} + C.$$

Solution 1.6. We can integrate by parts to recover the formula for the antiderivative,

$\pm$	D	I
$+$	$\operatorname{erf}(x)$	1
$-\int$	$\frac{d}{dx}\operatorname{erf}(x)$	x

Since $\frac{d}{dx}\text{erf}(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$ by the fundamental theorem, we have

$$\int \text{erf}(x)\,dx = x\text{erf}(x) - \int \frac{2x}{\sqrt{\pi}}e^{-x^2}\,dx.$$

The second integral can be solved using the substitution $u = -x^2$, $du = -2xdx$ which gives us

$$\int \text{erf}(x)\,dx = x\text{erf}(x) + \int \frac{1}{\sqrt{\pi}}e^{u}\,du = x\text{erf}(x) + \frac{1}{\sqrt{\pi}} \cdot e^{-x^2} + C$$

Remark. It is easy to check that the $x\text{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + C$ is an antiderivative by simply differentiating.

Problem 1.7. ($\star\star\star$) Using the integral definition of the error function, show that

$$\int_0^x e^{at} \cdot e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} \cdot e^{\frac{a^2}{4}} \cdot \Big(\text{erf}\Big(x - \frac{a}{2}\Big) + \text{erf}\Big(\frac{a}{2}\Big)\Big).$$

Solution 1.7. We want to write the integral in terms of the error function. The trick is to "complete the square" in the exponent

$$\begin{aligned}
\int_0^x e^{at} \cdot e^{-t^2}\,dt &= \int_0^x e^{-t^2+at-\frac{a^2}{4}+\frac{a^2}{4}}\,dt && \text{(complete the square)}\\
&= e^{\frac{a^2}{4}} \int_0^x e^{-(t-\frac{a}{2})^2}\,dt\\
&= e^{\frac{a^2}{4}} \int_{-\frac{a}{2}}^{x-\frac{a}{2}} e^{-u^2}\,du && u = t - \frac{a}{2},\ du = dt,\ \int_0^x dt \to \int_{-\frac{a}{2}}^{x-\frac{a}{2}} du\\
&= e^{\frac{a^2}{4}} \Big(\int_{-\frac{a}{2}}^{0} e^{-u^2}\,du + \int_0^{x-\frac{a}{2}} e^{-u^2}\,du\Big) && \int_a^b f(t)\,dt = \int_a^0 f(t)\,dt + \int_0^b f(t)\,dt\\
&= e^{\frac{a^2}{4}} \Big(-\int_0^{-\frac{a}{2}} e^{-u^2}\,du + \int_0^{x-\frac{a}{2}} e^{-u^2}\,du\Big) && \int_a^0 f(t)\,dt = -\int_0^a f(t)\,dt\\
&= \frac{\sqrt{\pi}}{2} \cdot e^{\frac{a^2}{4}} \cdot \Big(\text{erf}\Big(x - \frac{a}{2}\Big) - \text{erf}\Big(-\frac{a}{2}\Big)\Big) && \int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\text{erf}(x)\\
&= \frac{\sqrt{\pi}}{2} \cdot e^{\frac{a^2}{4}} \cdot \Big(\text{erf}\Big(x - \frac{a}{2}\Big) + \text{erf}\Big(\frac{a}{2}\Big)\Big). && (\text{erf}(x) \text{ is odd})
\end{aligned}$$

1.4.4 Curve Sketching

Strategy: Given $f(t)$ we want to sketch $F(x) = \int_0^x f(t)\,dt$. In general, it is very hard to find the values of $F(x)$ without using numerical integration methods, but we are able to do a rough sketch without any computations at all. Since

$$F'(x) = f(x) \quad \text{and} \quad F''(x) = f'(x)$$

we have the following rules:

$f(x)$	$F(x)$
Positive	Increasing
Negative	Decreasing
Increasing	Convex
Decreasing	Concave

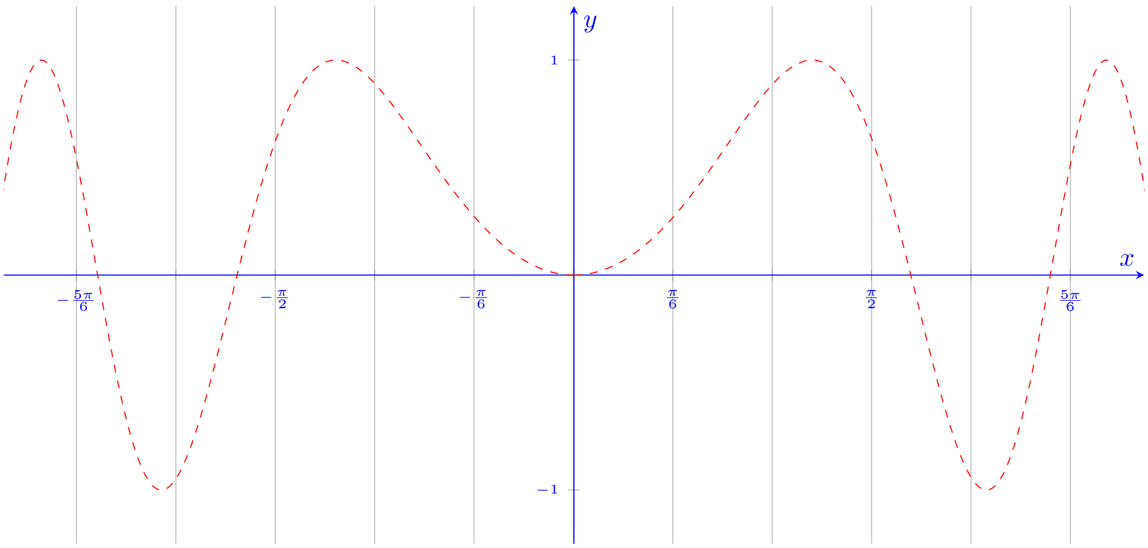
and

$f(x)$	$F(x)$
x-intercept (crossing)	local max/min
local max/min	inflection point

(1)

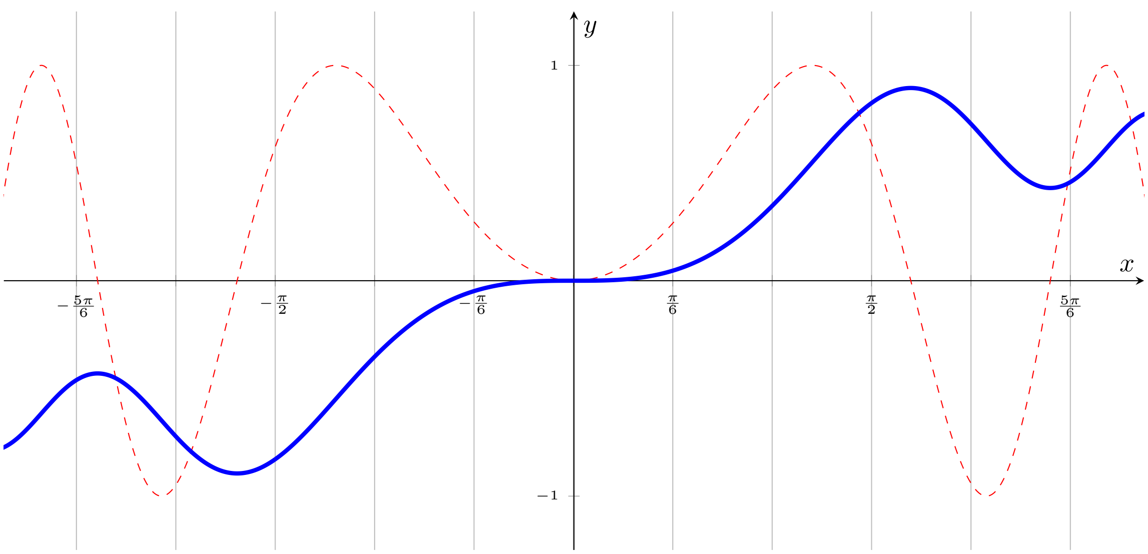
We also know that $F(0) = \int_0^0 f(t)\,dt = 0$. If $f(t)$ is odd or even, then we can use symmetry to recover the shape of $F(x)$ by doing an even or odd reflection of the graph of $F(x)$ around the y-axis. This means we only have to sketch $F(x)$ for $x \geq 0$, if $f(t)$ is symmetric.

Problem 1.8. ($\star\star$) The graph of $f(x) = \sin(x^2)$ is displayed as a dashed red line below:



Sketch the graph of the *Fresnel sine function* $S(x) = \int_0^x f(t)\,dt$.

Solution 1.8. We can use the rules in Table 1 to sketch the graph of $S(x)$ (displayed in blue)



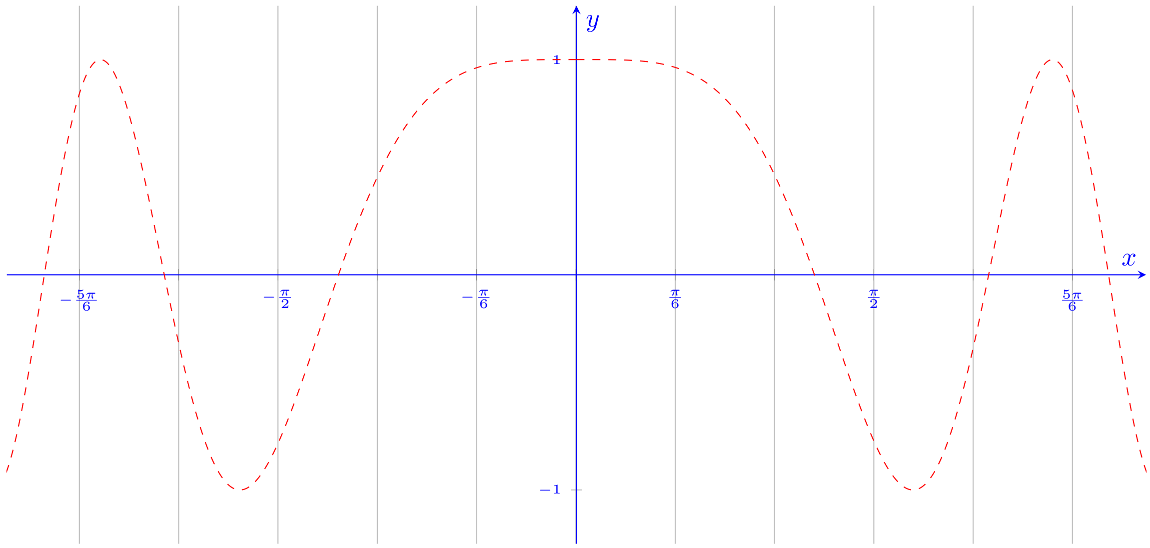
(a) $S(0) = 0$, so the graph passes through the origin.

(b) $S(x)$ is odd since $f(x) = \sin(x^2)$ is an even function and the lower limit of integration is 0.

(c) The maximum of $S(x)$ can be approximated by a triangle with width $\sqrt{\pi}$ and height 1,

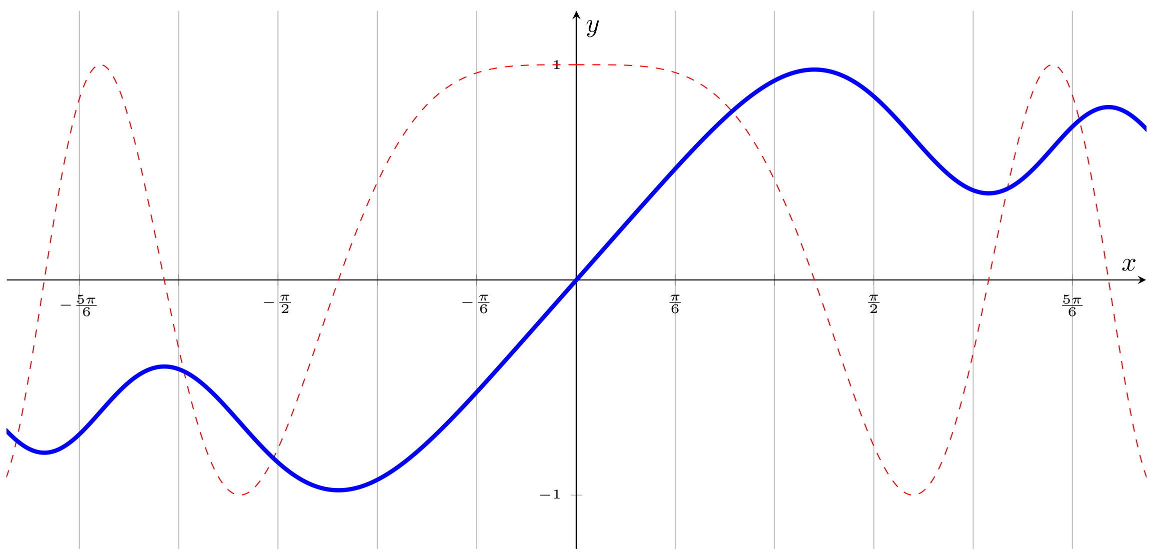
$$\int_0^{\sqrt{\pi}} \sin(t^2)\,dt \approx \frac{\sqrt{\pi}}{2} \approx 0.89.$$

Problem 1.9. ($\star\star$) The graph of $f(x) = \cos(x^2)$ is displayed as a dashed red line below:



Sketch the graph of the *Fresnel cosine function* $C(x) = \int_0^x f(t)\,dt$.

Solution 1.9. We can use the rules in Table 1 to sketch the graph of $C(x)$ (displayed in blue)



(a) $C(0) = 0$, so the graph passes through the origin.

(b) $C(x)$ is odd since $f(x) = \cos(x^2)$ is an even function and the lower limit of integration is 0.

(c) The maximum of $C(x)$ can be approximated using a midpoint Riemann sum with one rectangle,

$$\int_0^{\sqrt{\frac{\pi}{2}}} \cos(t^2)\,dt \approx \cos\left(\left(\sqrt{\frac{\pi}{2}} \cdot \frac{1}{2}\right)^2\right) \cdot \sqrt{\frac{\pi}{2}} \approx 1.16.$$