1 Using Integration to Find Areas

Formula: The area between the two curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is given by

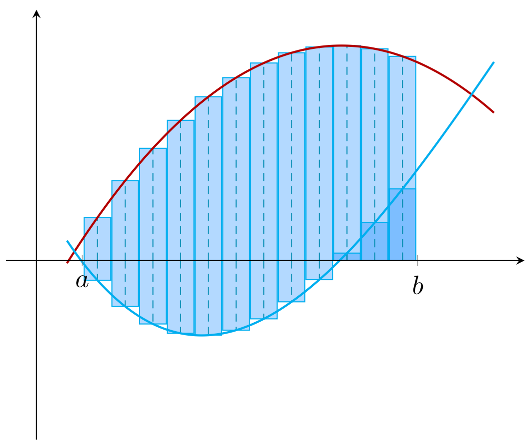
$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Intuition: We can approximate the area with small rectangles of the form

$$A_i = |f(x_i^*) - g(x_i^*)|\Delta x, \tag{1}$$

where x_i^* is a point in a subinterval of length Δx. If we partition $[a, b]$ into n uniform subintervals and approximate the area with rectangles of the form (1), taking the limit as $n \to \infty$ implies

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)|\Delta x = \int_a^b |f(x) - g(x)|\, dx.$$



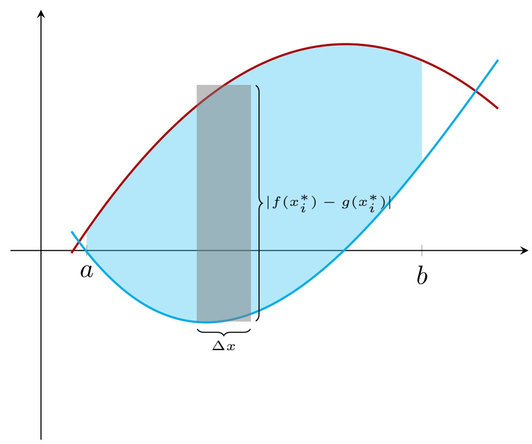


Figure: The height of the subrectangle is the distance between the two functions f and g. The area of each of the approximating rectangles is given by

$$\text{length} \times \text{width} = |f(x_i^*) - g(x_i^*)|\Delta x.$$

1.1 Total Distance Traveled

Net Distance: Let $v(t)$ be the velocity of a particle. The *net distance traveled* by the particle over the time interval $[a, b]$ is given by

$$\int_a^b v(t)\, dt.$$

The *average velocity* is given by

$$\frac{1}{b-a}\int_a^b v(t)\, dt.$$

Total Distance: Let $v(t)$ be the velocity of a particle. The *total distance traveled* by the particle over the time interval $[a, b]$ is given by

$$\int_a^b |v(t)|\, dt.$$

The *average speed* is given by

$$\frac{1}{b-a}\int_a^b |v(t)|\, dt.$$

1.2 Example Problems

1.2.1 Distance/Displacement Problems

Problem 1.1. ($\star\star$) Let $v(t) = 1 - \ln(1+t)$ be the speed of a particle for $0 \le t \le 5$.

1. Find the average velocity of the particle.
2. Find the average speed of the particle.

Solution 1.1.

Part (a) *Average Velocity:* The net distance is traveled is given by,

$$\begin{aligned}
\int_0^5 (1 - \ln(1+t))\,dt &= t - t\ln(1+t)\Big|_{t=0}^{t=5} + \int_0^5 \frac{t}{1+t}\,dt && \text{integration by parts} \\
&= t - t\ln(1+t)\Big|_{t=0}^{t=5} + \int_0^5 1 - \frac{1}{1+t}\,dt && \text{long division} \\
&= t - t\ln(1+t) + t - \ln(1+t)\Big|_{t=0}^{t=5} \\
&= 10 - 6\ln(6) \approx -0.7506.
\end{aligned}$$

The average velocity is therefore,

$$\frac{1}{5}\int_0^5 (1 - \ln(1+t))\,dt = \frac{1}{5}\Big(10 - 6\ln(6)\Big) \approx -0.15.$$

Part (b) *Average Speed:* The total distance traveled is given by

$$\int_0^5 |1 - \ln(1+t)|\,dt.$$

We first classify the signs of $f(t) = 1 - \ln(1+t)$. The roots are given by

$$1 - \ln(1+t) = 0 \Rightarrow 1 + t = e \Rightarrow t = e - 1.$$

The signs are also given by

Therefore, the integral is given by

$$\begin{aligned}
\int_0^5 |1 - \ln(1+t)|\,dt &= \int_0^{e-1} (1 - \ln(1+t))\,dt - \int_{e-1}^5 (1 - \ln(1+t))\,dt && \text{definition of } |\cdot| \\
&= 2t - (1+t)\ln(1+t)\Big|_{t=0}^{t=e-1} - (2t - (1+t)\ln(1+t))\Big|_{t=e-1}^{t=5} && \text{same steps as Part(a)} \\
&= 2(e-1) - e\ln(e) - (10 - 6\ln(6) - 2(e-1) + e\ln(e)) \\
&= -14 + 2e + 6\ln(6) \approx 2.1871.
\end{aligned}$$

The average speed is therefore,

$$\frac{1}{5}\int_0^5 |1 - \ln(1+t)|\,dt = \frac{1}{5}\Big(-14 + 2e + 6\ln(6)\Big) \approx 0.437.$$

1.2.2 Areas Between Curves

Strategy: The areas between curves can be computed without drawing a picture.

1. *(Optional) Draw the Curves:* Draw the curves on the (x, y) plane.
2. *Set up the definite integral:* Find the functions that represents the curves and the domain of integration. It may be useful to treat our curves as a function of y instead of x in some examples.
3. *Write the absolute value as a piecewise function:* Find the regions where $f(x) - g(x) > 0$ and $f(x) - g(x) < 0$ and split the region of integration into the different regions.
4. Compute the integrals.

Problem 1.2. ($\star$) Find the area of the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

Solution 1.2.

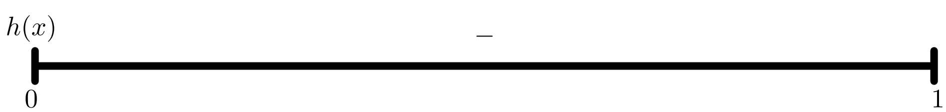
Finding the Integral: We first start by expressing the area as a definite integral. The first curve is given by $y = x^2$ and the second curve is given by $y = \sqrt{x}$. The curves intersect when

$$x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1.$$

The region of integration is given by the smallest and the largest of these values, so the area by

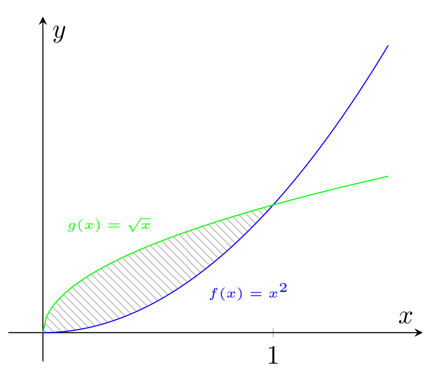
$$\int_0^1 |x^2 - \sqrt{x}| \, dx.$$

Compute the Integral: We first classify the signs of $h(x) = x^2 - \sqrt{x}$. From the first part, we found that the roots are given by $0, 1$ so the signs are given by



Therefore, the area is given by

$$\int_0^1 |x^2 - \sqrt{x}| \, dx = -\int_0^1 (x^2 - \sqrt{x}) \, dx = -\frac{x^3}{3} + \frac{2}{3}x^{3/2}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$



Problem 1.3. (⋆⋆) Find the area of the region bounded by the curves $y^2 + x = 1$ and $y^2 - x = 1$.

Solution 1.3. This problem is must easier to do if we treat x as a function of y.

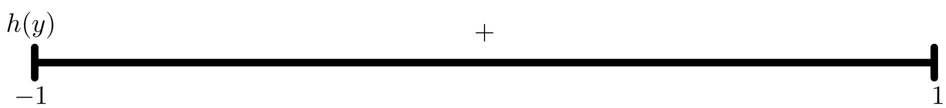
Finding the Integral: Our functions are given by $x = 1 - y^2$ and $x = y^2 - 1$. The curves intersect when

$$1 - y^2 = y^2 - 1 \Rightarrow 2y^2 - 2 = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = \pm 1.$$

Therefore, the integral is given by

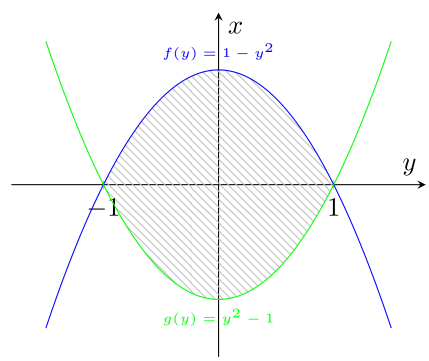
$$\int_{-1}^{1} |1 - y^2 - (y^2 - 1)|\, dy = \int_{-1}^{1} |2 - 2y^2|\, dy.$$

Compute the Integral: We first classify the signs of $h(y) = 2 - 2y^2$. From the first part, we found that the roots are given by $-1, 1$ so the signs are given by



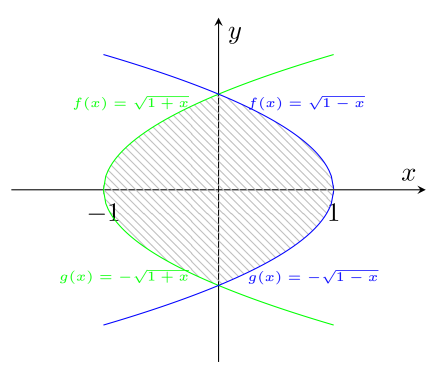
Therefore, the area is given by

$$\int_{-1}^{1} |2 - 2y^2|\, dy = \int_{-1}^{1} 2 - 2y^2\, dy = 2y - \frac{2}{3}y^3\Big|_{y=-1}^{y=1} = 4 - \frac{4}{3} = \frac{8}{3}.$$



Remark. If we integrated with respect to x, then we would have computed

$$\int_{-1}^{0} \sqrt{1+x} + \sqrt{1+x}\, dx + \int_{0}^{1} \sqrt{1-x} + \sqrt{1-x}\, dx = \frac{4}{3}(1+x)^{3/2}\Big|_{x=-1}^{x=0} - \frac{4}{3}(1-x)^{3/2}\Big|_{x=0}^{x=1} = \frac{8}{3}.$$



Problem 1.4. ($\star\star$) Determine the area enclosed by $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on $[0, 2\pi]$.

Solution 1.4.

Finding the Integral: The area enclosed by the curves $f(x)$ and $g(x)$ is

$$\int_0^{2\pi} |\sin(x) - \cos(x)|\, dx.$$

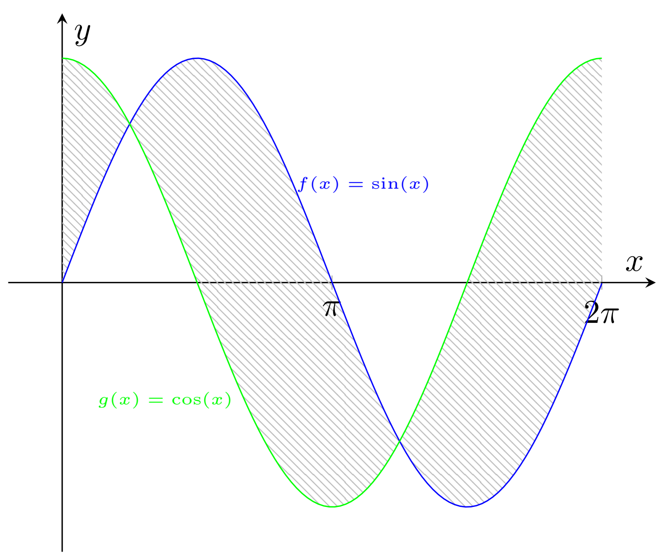
Compute the Integral: We first classify the signs of $h(x) = \sin(x) - \cos(x)$. The roots are given by

$$\sin(x) - \cos(x) = 0 \Rightarrow \tan(x) = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}.$$

The signs are also given by

Therefore, the integral is given by

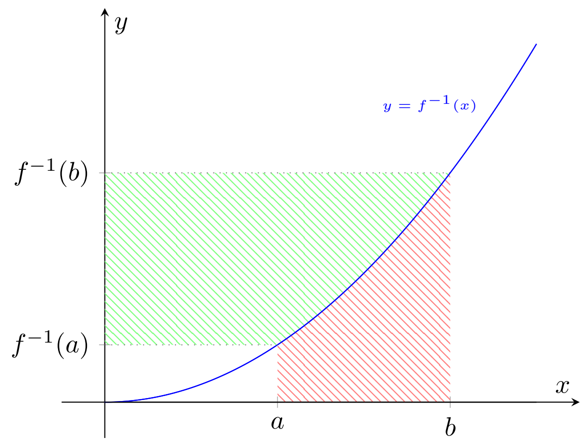
$$\begin{aligned}
&\int_0^{2\pi} |\sin(x) - \cos(x)|\, dx \\
&= -\int_0^{\frac{\pi}{4}} \sin(x) - \cos(x)\, dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x) - \cos(x)\, dx - \int_{\frac{5\pi}{4}}^{2\pi} \sin(x) - \cos(x)\, dx \\
&= -(-\cos(x) - \sin(x))\Big|_0^{\frac{\pi}{4}} + (-\cos(x) - \sin(x))\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - (-\cos(x) - \sin(x))\Big|_{\frac{5\pi}{4}}^{2\pi} \\
&= (\sqrt{2} - 1) + (\sqrt{2} + \sqrt{2}) + (1 + \sqrt{2}) \\
&= 4\sqrt{2}.
\end{aligned}$$



Problem 1.5. ($\star\star\star$) Prove the integral of the inverse function formula,

$$\int_a^b f^{-1}(x)\,dx = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)\,dx.$$

Solution 1.5. For simplicity, we assume that $0 \le a < b$ and $0 \le f^{-1}(a) < f^{-1}(b)$. Consider the following diagram:



The area in red represents $\int_a^b f^{-1}(x)\,dx$ and the area in green represents $\int_{f^{-1}(a)}^{f^{-1}(b)} f(y)\,dy$. The total area of the shaded region is given by $bf^{-1}(b) - af^{-1}(a)$. Therefore, the picture suggests that

$$\int_a^b f^{-1}(x)\,dx + \int_{f^{-1}(a)}^{f^{-1}(b)} f(y)\,dy = bf^{-1}(b) - af^{-1}(a).$$

Rearranging terms and changing the index of integration gives us the formula,

$$\int_a^b f^{-1}(x)\,dx = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(x)\,dx.$$

A similar diagram can be drawn without assumptions on $a, b, f^{-1}(a), f^{-1}(b)$ to get the general result.

Remark. The indefinite integral version of the formula is given by

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F(f^{-1}(x)) + C,$$

where $F(x)$ is an antiderivative of $f(x)$. The formula can be verified by differentiating the right hand side. We can also see that this agrees with the definite integral version proved above if we include the bounds of integration.

Remark. The formula is useful because it expresses the antiderivative of the inverse function in terms of the original function. For example, the formula implies that

$$\begin{aligned}\int_0^{1/2} \cos^{-1}(x)\,dx &= \frac{1}{2}\cdot\cos^{-1}\left(\frac{1}{2}\right) - 0\cdot\cos^{-1}(0) - \int_{\cos^{-1}(0)}^{\cos^{-1}(\frac{1}{2})} \cos(x)\,dx \\ &= \frac{\pi}{6} - \left(\sin(x)\Big|_{\pi/2}^{\pi/3}\right) \\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1.\end{aligned}$$

2 Using Integration to Find Volumes

2.1 Volumes Using Cross-Sectional Area

Formula: The *volume of a solid* with cross-sectional areas $A(x)$ perpendicular to the x-axis from $x = a$ to $x = b$ is

$$V = \int_a^b A(x)\,dx.$$

Intuition: We can approximate the volume with small cylinders of the form

$$V_i = A(x_i^*)\Delta x, \tag{2}$$

where x_i^* is a point in a subinterval of length Δx. If we partition $[a, b]$ into n uniform subintervals and approximate the area with cylinders of the form (2), taking the limit as $n \to \infty$ implies

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx.$$

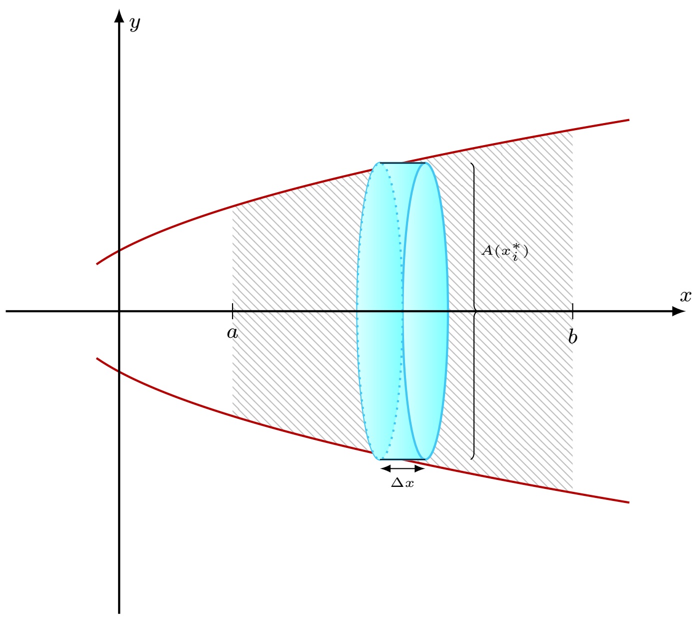


Figure: The base area of the cylinder is $A(x_i^*)$ and the height of the cylinder is Δx. The area of each of the approximating cylinders is given by

$$\text{base area} \times \text{height} = A(x_i^*)\Delta x.$$

2.2 Volumes Using Washers (Rotation around a Horizontal Axis)

Formula: The *volume of the solid of revolution* rotated about a *horizontal* axis with outer radius $R(x)$ and inner radius $r(x)$ from $x = a$ to $x = b$ is

$$V = \int_a^b \left(\pi R(x)^2 - \pi r(x)^2\right) dx.$$

Intuition: This formula is a special case of the volumes using cross-sectional area when the cross-sectional area of the solid is a annulus with inner $r(x)$ and outer radius $R(x)$. The cross sectional area is given explicitly by

$$A(x) = \pi R(x)^2 - \pi r(x)^2.$$

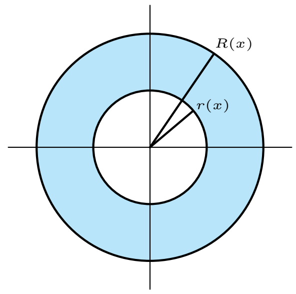


Figure: The cross-sectional area of a solid generated by rotation around a horizontal axis is given by

$$\text{Area Outer Circle} - \text{Area Inner Circle} = \pi R(x)^2 - \pi r(x)^2.$$

2.3 Volumes Using Shells (Rotation around a Vertical Axis)

Formula: The *volume of the solid of revolution* rotated about a *vertical* axis with upper height $H(x)$ and lower height $h(x)$ from $x = a$ to $x = b$ at a (positive) distance $r(x)$ from the axis of revolution is

$$V = \int_a^b 2\pi r(x)\big(H(x) - h(x)\big)\,dx.$$

Intuition: We can approximate the volume with small cylinders of the form

$$V_i = 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x = 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x, \tag{3}$$

where x_i^* is a point in a subinterval of length Δx. If we partition $[a, b]$ into n uniform subintervals and approximate the area with cylinders of the form (3), taking the limit as $n \to \infty$ implies

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x = \int_a^b 2\pi r(x)\big(H(x) - h(x)\big)\,dx.$$

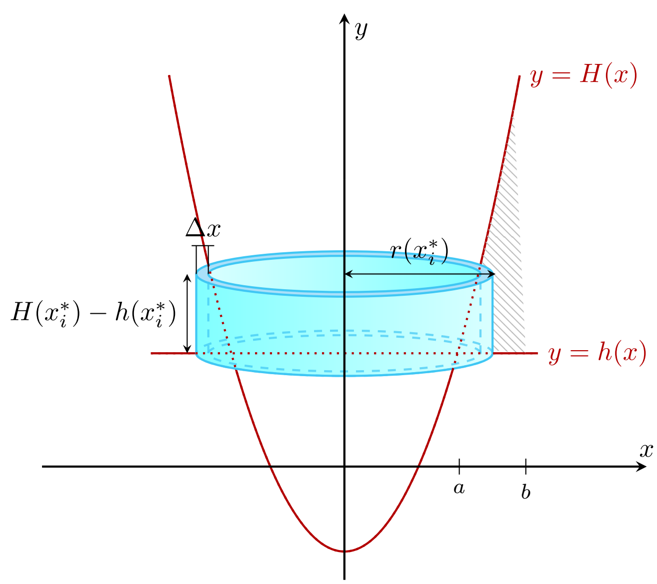


Figure: The length of the cylindrical shell is given by the radius of a circle, length $= 2\pi r(x_i^*)$. The area of each of the approximating cylindrical shells is given by

$$\text{length} \times \text{height} \times \text{width} = 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x.$$

Remark. If the rotation is about the y-axis, and $0 \le a < b$ (the region is to the right of the axis of rotation), then the radius $r(x) = x$ and the formula is $V = \int_a^b 2\pi x\big(H(x) - h(x)\big)\,dx$.

2.4 Example Problems:

Strategy:

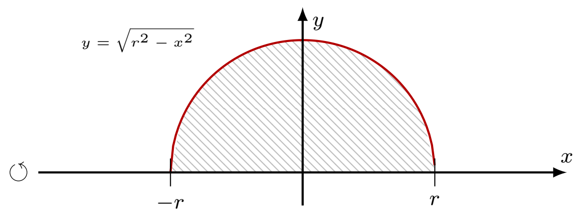
1. *(Optional) Draw the Projected Area:* Draw the area of the curve projected onto the (x, y) plane.
2. *Set up the definite integral:* Find a formula for volume using either the cross sectional area or cylindrical shells. Choose the representation that will result in a simpler integral.
3. Compute the integral.

Remark. Sometimes it might be more convenient to integrate with respect to y instead of x. All the formulas in this section can be easily modified to by interchanging the axes (replace the x variable with a y variable and treat all functions as functions of y instead of x).

Problem 2.1. $(\star)$ Compute the volume of a ball with radius r.

Solution 2.1. The area can be computed using either washers or cylindrical shells.

Washers: The ball is generated by rotating the area under the curve of $y = \sqrt{r^2 - x^2}$ on the interval $[-r, r]$ around the x-axis.



Finding the Integral: The cross-sectional area is a circle with radius $\sqrt{r^2 - x^2}$. Therefore, the cross-sectional area is given by

$$A(x) = \pi(r^2 - x^2).$$

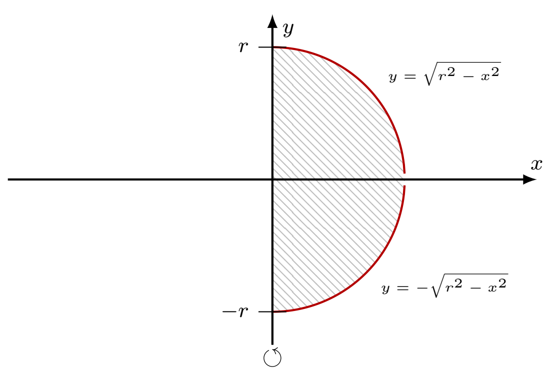
Using the volume formula for washers, the volume integral is

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx.$$

Computing the Integral: The integrand is even, so

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx = 2\int_{0}^{r} \pi(r^2 - x^2)\, dx = 2\pi r^2 x - 2\pi \frac{x^3}{3}\Big|_{x=0}^{x=r} = \frac{4}{3}\pi r^3.$$

Cylindrical Shells: The ball is generated by rotating the area bounded by the curves $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$ on the interval $[0, r]$ around the y-axis.



Finding the Integral: The radius of the cylinder is given by $r(x) = x$ and the height of the cylinder is given by

$$H(x) - h(x) = \sqrt{r^2 - x^2} - (-\sqrt{r^2 - x^2}) = 2\sqrt{r^2 - x^2}.$$

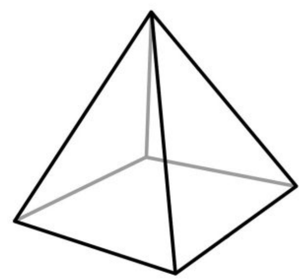
Using the volume formula for cylindrical shells, the volume integral is

$$\int_0^r 2\pi x(2\sqrt{r^2 - x^2})\,dx = \int_0^r 4\pi x\sqrt{r^2 - x^2}\,dx.$$

Computing the Integral: Using the substitution $u = r^2 - x^2$, we have

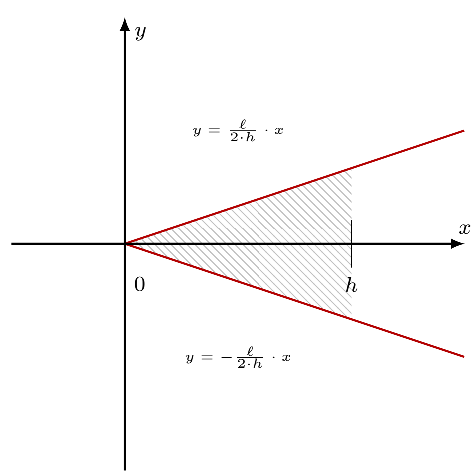
$$\int_0^r 4\pi x\sqrt{r^2 - x^2}\,dx = -2\pi \cdot \frac{2}{3}(r^2 - x^2)^{3/2}\Big|_{x=0}^{x=r} = \frac{4}{3}\pi r^3.$$

Problem 2.2. ($\star$) Compute the volume of square pyramid with base length ℓ and height h.



Solution 2.2. The area can be computed using cross sectional area.

Cross-Sectional Area: We will orient the pyramid along the x-axis with vertex at the origin. The upper edge of the pyramid must pass through the point $(h, \ell/2)$, so the height is given by $y = \frac{\ell}{2h} \cdot x$. Similarly the height of the lower edge of the pyramid is given by $y = -\frac{\ell}{2h} \cdot x$.



Finding the Integral: The cross-sectional area is a square with side length $2 \cdot \frac{\ell}{2h}x$. Therefore, the cross sectional-area is given by

$$A(x) = \left(2 \cdot \frac{\ell}{2h}x\right)^2 = \frac{\ell^2}{h^2}x^2.$$

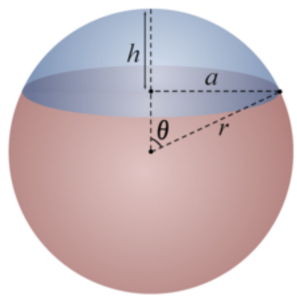
Using the volume formula for cross-sections, the volume integral is

$$\int_0^h \frac{\ell^2}{h^2}x^2\,dx.$$

Computing the Integral: This integral is easy to compute,

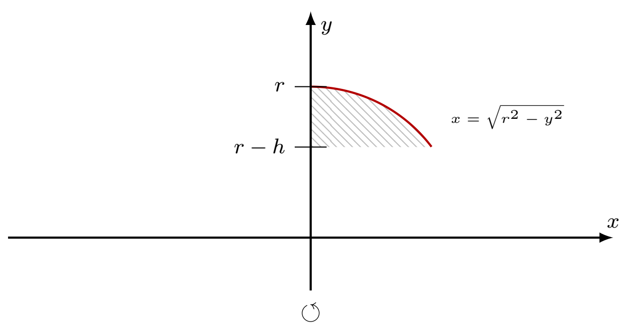
$$\int_0^h \frac{\ell^2}{h^2}x^2\,dx = \frac{\ell^2}{3h^2}x^3\Big|_{x=0}^{x=h} = \frac{\ell^2}{3h^2}h^3 = \frac{1}{3}\ell^2 h.$$

Problem 2.3. (⋆⋆) Compute the volume of a spherical cap with height $h < r$ from a ball radius r.



Solution 2.3. The volume can be computed using either washers or cylindrical shells. The method with cylindrical shells is a bit harder in this case.

Washers: The spherical cap is generated by rotating the area under the curve of $x = \sqrt{r^2 - y^2}$ on the interval $[r-h, r]$ around the y-axis.



Finding the Integral: The cross-sectional area is a circle with radius $\sqrt{r^2-y^2}$. Therefore, the cross-sectional area is given by

$$A(y) = \pi(r^2 - y^2).$$

Using the volume formula for washers, the volume integral is

$$\int_{r-h}^{r} \pi(r^2 - y^2)\,dy.$$

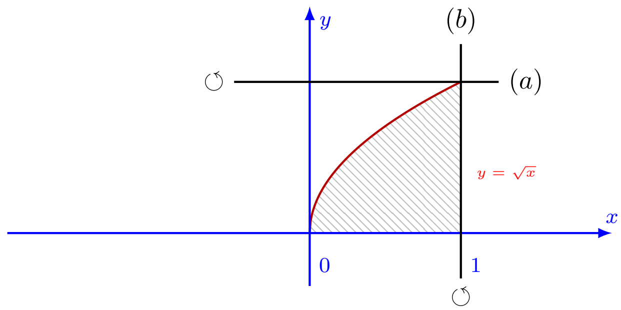
Computing the Integral: This integral is easy to compute,

$$\int_{r-h}^{r} \pi(r^2-y^2)\,dy = \pi r^2 y - \frac{\pi}{3}y^3\Big|_{y=r-h}^{y=r} = \frac{1}{3}\pi h^2(3r-h).$$

Problem 2.4. ($\star\star$) Find the volume of the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 1$ rotated around

(a) the line $y = 1$

(b) the line $x = 1$.



Solution 2.4.

(a) **Washers:** We will compute the first volume using a washer.

Finding the Integral: The region of integration is $x \in [0, 1]$, the inner radius is $r(x) = 1 - \sqrt{x}$ and the outer radius is $R(x) = 1$. Therefore, the cross-sectional area is given by

$$A(x) = \pi(1^2 - (1 - \sqrt{x})^2) = 2\pi\sqrt{x} - \pi x.$$

Using the volume formula for washers, the volume integral is

$$\int_0^1 2\pi\sqrt{x} - \pi x \, dx.$$

Computing the Integral: This integral is easy to compute,

$$\int_0^1 2\pi\sqrt{x} - \pi x \, dx = 2\pi \frac{2}{3} x^{3/2} - \pi \frac{x^2}{2} \Big|_{x=0}^{x=1} = \frac{5\pi}{6}.$$

Remark. If we integrated with respect to y using the cylindrical shell formula, we would get

$$\int_0^1 2\pi(1-y)(1-y^2)\, dy = 2\pi \int_0^1 1 - y - y^2 + y^3 \, dy = 2\pi\Big(y - \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4}\Big)\Big|_{y=0}^{y=1} = \frac{5\pi}{6}.$$

(b) **Shells:** We will compute the second volume using a cylindrical shell.

Finding the Integral: The region of integration is $x \in [0, 1]$, the radius is $r(x) = 1 - x$ and the height is $H(x) = \sqrt{x}$. Using the volume formula for shells, the volume integral is

$$\int_0^1 2\pi(1-x)\sqrt{x}\, dx.$$

Computing the Integral: This is easy to compute,

$$\int_0^1 2\pi(1-x)\sqrt{x}\, dx = 2\pi \cdot \frac{2}{3} x^{3/2} - 2\pi \cdot \frac{2}{5} x^{5/2} \Big|_{x=0}^{x=1} = \frac{8\pi}{15}.$$

Remark. If we integrated with respect to y using the washer formula, we would get

$$\int_0^1 \pi(1-y^2)^2\, dy = \pi \int_0^1 1 - 2y^2 + y^4 \, dy = \pi \cdot \Big(y - \frac{2}{3}y^3 + \frac{y^5}{5}\Big)\Big|_{y=0}^{y=1} = \frac{8\pi}{15}.$$