1 Integration By Parts

We can think of integration by substitution as the counterpart of the product rule for differentiation. Suppose that $u(x)$ and $v(x)$ are continuously differentiable functions. Integration by parts is given by the following formulas:

Indefinite Integral Version:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

Definite Integral Version:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_{x=a}^{x=b} - \int_a^b u'(x)v(x)\,dx.$$

1.1 Tabular Method

We will introduce a method to bookkeep multiple integration by parts steps simultaneously. This is called the tabular method for integration by parts. You pick a term to differentiate and a term to integrate then repeat the operation until the product of the terms in the last entry of the table is easy to integrate.

The integral can be recovered by multiplying diagonally across the rows of the table adding up all terms with alternating signs. The last term in the table is integrated across.

For example, the formula to integrate $\int u(x)v'''(x)\,dx$ by parts can be encoded by the table

$\pm$	D	I
$+$	u	v'''
$-$	u'	v''
$+$	u''	v'
$-\int$	u'''	v

which gives us the formula

$$\int u(x)v'''(x)\,dx = u(x)v''(x) - u'(x)v'(x) + u''(x)v(x) - \int u'''(x)v(x)\,dx.$$

1.2 Example Problems

Problem 1.1. ($\star$) Find

$$\int xe^x\,dx.$$

Solution 1.1.

Step 1: Draw the table

$\pm$	D	I
$+$	x	e^x
$-$	1	e^x
$+\int$	0	e^x

Step 2: From the table, we have

$$\int xe^x\,dx = xe^x - e^x + C.$$

Problem 1.2. ($\star$) Find

$$\int \ln(x)\,dx.$$

Solution 1.2.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln(x)$	1
$-\int$	$\frac{1}{x}$	x

Step 2: From the table, we have

$$\int \ln(x)\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C.$$

Problem 1.3. ($\star\star$) Find

$$\int x^4\sin(x)\,dx.$$

Solution 1.3.

Step 1: Draw the table

$\pm$	D	I
$+$	x^4	$\sin x$
$-$	$4x^3$	$-\cos x$
$+$	$12x^2$	$-\sin x$
$-$	$24x$	$\cos x$
$+$	24	$\sin x$
$-\int$	0	$-\cos x$

Step 2: From the table, we have

$$\int x^4\sin x\,dx = -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + C.$$

Problem 1.4. ($\star\star$) Find

$$\int e^x \sin(x)\, dx.$$

Solution 1.4.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin x$	e^x
$-$	$\cos x$	e^x
$+\int$	$-\sin x$	e^x

Step 2: From the table, we have

$$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx + D.$$

Moving all the $\int e^x \sin x\, dx$ to one side and simplifying, we can conclude

$$2\int e^x \sin x\, dx = e^x \sin x - e^x \cos x + D \implies \int e^x \sin x\, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C.$$

Problem 1.5. ($\star\star\star$) Find

$$\int x e^x \cos(x)\, dx.$$

Solution 1.5.

Step 1: Draw the table

$\pm$	D	I
$+$	$x \cos x$	e^x
$-$	$\cos x - x \sin x$	e^x
$+\int$	$-2\sin x - x\cos x$	e^x

Step 2: From the table, we have

$$\int x e^x \cos x\, dx = x e^x \cos x - e^x \cos x + x e^x \sin x - 2\int e^x \sin x\, dx - \int x e^x \cos x\, dx.$$

Moving all the $\int x e^x \cos x\, dx$ to one side and simplifying, we can conclude

$$\begin{aligned} 2\int x e^x \cos x\, dx &= x e^x \cos x - e^x \cos x + x e^x \sin x - 2\int e^x \sin x\, dx \\ &= x e^x \cos x - e^x \cos x + x e^x \sin x - e^x \sin x + e^x \cos x + C. \qquad \text{Problem 4} \end{aligned}$$

Dividing both sides by 2, we can conclude

$$\int x e^x \cos x\, dx = \frac{1}{2}\Big(x e^x \cos x + x e^x \sin x - e^x \sin x \Big) + C.$$

Problem 1.6. (⋆⋆) Evaluate

$$\int_1^2 x^3 \ln x \, dx.$$

Solution 1.6.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln x$	x^3
$-\int$	$\frac{1}{x}$	$\frac{1}{4}x^4$

Step 2: From the table, we have

$$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$$

Step 3: We can now use the fundamental theorem of calculus to compute the definite integral,

$$\int_1^2 x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \Big|_{x=1}^{x=2} = 4\ln 2 - 1 + \frac{1}{16} = 4\ln 2 - \frac{15}{16}.$$

Problem 1.7. (⋆⋆⋆) Derive the reduction formula

$$\int \sin^n(x)\, dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx.$$

Solution 1.7.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin^{n-1}(x)$	$\sin(x)$
$-\int$	$(n-1)\cos(x)\sin^{n-2}(x)$	$-\cos(x)$

Step 2: From the table, we have

$$\begin{aligned}\int \sin^n(x)\, dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x) \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x) && \sin^2(x)+\cos^2(x)=1 \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\, dx - (n-1)\int \sin^n(x)\, dx\end{aligned}$$

Moving all the the $\int \sin^n(x)\, dx$ terms to one side, we can conclude

$$\begin{aligned} n\int \sin^n(x)\, dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\, dx \\ \Rightarrow \int \sin^n(x)\, dx &= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx.\end{aligned}$$

Problem 1.8. ($\star\star$) Find

$$\int \sin(\ln(x))\,dx.$$

Solution 1.8. This problem can be solved integrating by parts twice.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin(\ln(x))$	1
$-\int$	$\frac{1}{x}\cdot\cos(\ln(x))$	x

Step 2: From the table, we have

$$\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - \int \cos(\ln(x))\,dx.$$

To proceed further, we have to integrate the second term by parts.

Step 3: Draw the table

$\pm$	D	I
$+$	$\cos(\ln(x))$	1
$-\int$	$-\frac{1}{x}\cdot\sin(\ln(x))$	x

Step 4: From the table, we have

$$\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - \left(x\cos(\ln(x)) + \int \sin(\ln(x))\,dx\right).$$

Moving all the $\int \sin(\ln(x))\,dx$ terms to one side and simplifying, we can conclude

$$\int \sin(\ln(x))\,dx = \frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x)) + C.$$

Remark: This problem can also be solved using the change of variables $u = \ln(x)$. Under this change of variables, the integral reduces to

$$\int \sin(\ln(x))\,dx = \int e^u \sin(u)\,du.$$

This integral can now be solved by integrating by parts twice similarly to Problem 1.4.

2 Integration By Substitution (Change of Variables)

We can think of integration by substitution as the counterpart of the chain rule for differentiation. Suppose that $g(x)$ is a differentiable function and f is continuous on the range of g. Integration by substitution is given by the following formulas:

Indefinite Integral Version:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \qquad \text{where } u = g(x).$$

Definite Integral Version:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad \text{where } u = g(x).$$

2.1 Example Problems

Strategy: The idea is to make the integral easier to compute by doing a change of variables.

1. Start by guessing what the appropriate change of variable $u = g(x)$ should be. Usually you choose u to be the function that is "inside" the function.
2. Differentiate both sides of $u = g(x)$ to conclude $du = g'(x)dx$. If we have a definite integral, use the fact that $x = a \to u = g(a)$ and $x = b \to u = g(b)$ to also change the bounds of integration.
3. Rewrite the integral by replacing all instances of x with the new variable and compute the integral or definite integral.
4. If you computed the indefinite integral, then make sure to write your final answer back in terms of the original variables.

Problem 2.1. ($\star$) Evaluate

$$\int_0^1 x\sqrt{1-x^2}\,dx.$$

Solution 2.1.

Step 1: We will use the change of variables $u = 1 - x^2$,

$$\frac{du}{dx} = -2x \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = xdx, \qquad x = 0 \to u = 1, \qquad x = 1 \to u = 0.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=0} = \frac{1}{3}.$$

Remark: Instead of changing the bounds of integration, we can first find the indefinite integral,

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}},$$

then use the fundamental theorem of calculus to conclude

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

Problem 2.2. ($\star$) Find
$$\int \tan(x)\,dx.$$

Solution 2.2.

Step 1: We will use the change of variables $u = \cos(x)$,
$$\frac{du}{dx} = -\sin(x) \Rightarrow du = -\sin(x)\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,
$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos(x)| + C.$$

Problem 2.3. ($\star$) Find
$$\int \tanh(x)\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$$

Solution 2.3.

Step 1: We will use the change of variables $u = e^x + e^{-x}$,
$$\frac{du}{dx} = e^x - e^{-x} \Rightarrow du = (e^x - e^{-x})\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,
$$\begin{aligned}\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{du}{u} &= \ln|u| + C \\ &= \ln|e^x + e^{-x}| + C. \qquad u = e^x + e^{-x}\end{aligned}$$

Since $e^x + e^{-x} > 0$, we can remove the absolute values if we wish giving the final answer
$$\int \tanh(x)\,dx = \ln(e^x + e^{-x}) + C.$$

Remark: We can use the fact $e^x + e^{-x} = 2\cosh(x)$ to conclude that
$$\ln(e^x + e^{-x}) + C = \ln(2\cosh(x)) + C = \ln(\cosh(x)) + \underbrace{\ln(2) + C}_{D} = \ln(\cosh(x)) + D.$$

This form of the indefinite integral may be easier to remember since it mirrors the fact that
$$\int \tan(x)\,dx = -\ln|\cos(x)| + C.$$

Problem 2.4. ($\star$) Find
$$\int \frac{1}{4+x^2}\,dx.$$

Solution 2.4.

We first factor out the 4 in the denominator,
$$\int \frac{1}{4+x^2}\,dx = \frac{1}{4}\int \frac{1}{1+\frac{x^2}{4}}\,dx = \frac{1}{4}\int \frac{1}{1+(\frac{x}{2})^2}\,dx.$$
Step 1: We will use the change of variables $u = \frac{x}{2}$,
$$\frac{du}{dx} = \frac{1}{2} \Rightarrow 2du = dx.$$
Step 2: We can now evaluate the integral under this change of variables,
$$\int \frac{1}{4+x^2}\,dx = \frac{1}{4}\int \frac{1}{1+(\frac{x}{2})^2}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + C = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Problem 2.5. ($\star\star$) Find
$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx.$$

Solution 2.5.

Step 1: We will use the change of variables $u = e^x$,
$$\frac{du}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}\,du \Rightarrow dx = \frac{1}{u}\,du.$$
Step 2: We can now evaluate the integral under this change of variables,
$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx &= \int \frac{2}{u(u+u^{-1})}\,du \\ &= \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u) + C \\ &= 2\tan^{-1}(e^x) + C. \qquad u = e^x\end{aligned}$$

Alternative Solution: We first do a trick by multiplying the numerator and denominator by e^x,
$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx.$$
Step 1: We will use the change of variables $u = e^x$,
$$\frac{du}{dx} = e^x \Rightarrow du = e^x\,dx.$$
Step 2: We can now evaluate the integral under this change of variables,
$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx &= \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u) + C \\ &= 2\tan^{-1}(e^x) + C. \qquad u = e^x\end{aligned}$$

Problem 2.6. ($\star\star$) Find

$$\int \sec(x)\,dx.$$

Solution 2.6. We first do a trick by multiplying the numerator and denominator by $\sec(x)+\tan(x)$,

$$\int \sec(x)\,dx = \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx.$$

Step 1: We will use the change of variables $u = \sec(x)+\tan(x)$,

$$\frac{du}{dx} = \sec(x)\tan(x)+\sec^2(x) \Rightarrow du = (\sec(x)\tan(x)+\sec^2(x))\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \sec(x)\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x))}{\sec(x)+\tan(x)}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u|+C \\ &= \ln|\sec(x)+\tan(x)|+C. \qquad u=\sec(x)+\tan(x)\end{aligned}$$

Problem 2.7. ($\star\star$) Find

$$\int \frac{1}{1+\sqrt{x}}\,dx.$$

Solution 2.7.

Step 1: We will use the change of variables $u=\sqrt{x}$,

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x}du = dx \Rightarrow 2u\,du = dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+u}\,du.$$

This integral is a bit tricky to compute, so we have to use algebra to simplify it first. Using long division to first simplify the integrand, we get

$$\begin{aligned}\int \frac{2u}{1+u}\,du = 2\int \frac{u}{1+u}\,du &= 2\int 1-\frac{1}{1+u}\,du \\ &= 2u - 2\ln|1+u|+C \\ &= 2\sqrt{x}-2\ln|1+\sqrt{x}|+C. \qquad u=\sqrt{x}.\end{aligned}$$

Alternative Solution: We can also do a change of variables by writing x as a function of u.

Step 1: We can also do the change of variables $x=u^2$,

$$\frac{dx}{du} = 2u \Rightarrow dx = 2u\,du.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+\sqrt{u^2}}\,du = \int \frac{2u}{1+u}\,du.$$

The computation is now identical to the case above.

2.1.1 Integrals Involving both Integration by Parts and Integration by Substitution

Problem 2.8. ($\star\star$) Find

$$\int_0^1 e^{\sqrt[4]{x}}\,dx.$$

Solution 2.8.

Step 1: Using the change of variables

$$x = u^4, \quad dx = 4u^3\,du, \quad \int_0^1 dx \to \int_0^1 du$$

we have

$$\int_0^1 e^{\sqrt[4]{x}}\,dx = 4\int_0^1 u^3 e^u\,du.$$

Step 2: We can now integrate by parts,

$\pm$	D	I
$+$	u^3	e^u
$-$	$3u^2$	e^u
$+$	$6u$	e^u
$-$	6	e^u
$+\int$	0	e^u

and therefore,

$$4\int_0^1 u^3 e^u\,du = 4\Big(u^3e^u - 3u^2e^u + 6ue^u - 6e^u\Big)\Big|_{u=0}^{u=1} = -8e + 24.$$

2.1.2 Proofs Involving a Change of Variables

Problem 2.9. ($\star\star\star$) Suppose that $f(-x) = f(x)$. Prove that

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Solution 2.9. By the properties of definite integrals, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$

Using the change of variables $u = -x$ on the first integral, for even function f,

$$\begin{aligned}\int_0^{-a} f(x)\,dx &= -\int_0^a f(-u)\,du && u=-x,\ du=-dx,\ x=0\to u=0,\ x=-a\to u=a\\ &= -\int_0^a f(u)\,du && f(-x)=f(x)\\ &= -\int_0^a f(x)\,dx.\end{aligned}$$

This computation implies

$$\int_{-a}^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Problem 2.10. ($\star\star\star$) Suppose that $f(-x) = -f(x)$. Prove that

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Solution 2.10. By the properties of definite integrals, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Using the change of variables $u = -x$ on the first integral, for odd functions f,

$$\begin{aligned}\int_{0}^{-a} f(x)\,dx &= -\int_{0}^{a} f(-u)\,du && u=-x,\ du=-dx,\ x=0\to u=0,\ x=-a\to u=a\\ &= \int_{0}^{a} f(u)\,du && f(-x)=-f(x)\\ &= \int_{0}^{a} f(x)\,dx.\end{aligned}$$

This computation implies

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

Problem 2.11. ($\star\star\star$) Justify the technique used to solve separable ordinary differential equations:

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx \Rightarrow G(y) = F(x) + C$$

where $G(y)$ is an antiderivative of $\frac{1}{g(y)}$ and $F(x)$ is an antiderivative of $f(x)$.

Solution 2.11. Using the notation $y'(x) = \frac{dy}{dx}$ and writing $y = y(x)$ explicitly as a function of x, we have

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{y'(x)}{g(y(x))} = f(x) \Rightarrow \int \frac{y'(x)}{g(y(x))}\,dx = \int f(x)\,dx.$$

Using the change of variables $u = y(x)$ on the first integral involving the $y(x)$ term, we see

$$\int \frac{y'(x)}{g(y(x))}\,dx = \int \frac{du}{g(u)} = \int \frac{dy}{g(y)}.$$

Therefore, using this change of variables, we can conclude that

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx.$$

This means there is a hidden change of variables that goes on when we formally separated $\frac{dy}{dx}$ in the second step of the technique.

3 Average Value of a Function

The average value of a function f on the interval $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We can interpret f_{ave} as the number such that half the area of the curve lies above it, and half the area lies below it. In other words, f_{ave} is the number such that

$$\int_a^b (f(x) - f_{\text{ave}})\,dx = 0.$$

Example: The average of $f(x) = x^2$ and $g(x) = x + 1$ on the interval $[-2, 2]$ is displayed below:

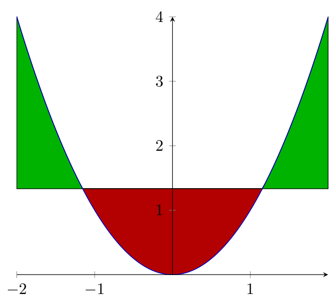


Figure 1: $f_{\text{ave}} = \frac{1}{4}\int_{-2}^{2} x^2\,dx = \frac{4}{3}$

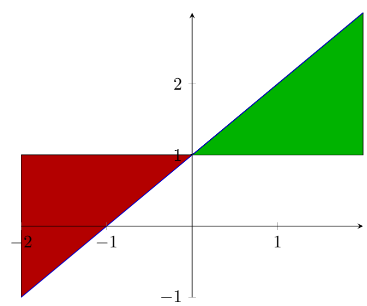


Figure 2: $g_{\text{ave}} = \frac{1}{4}\int_{-2}^{2} x + 1\,dx = 1$

3.1 More Properties of Definite Integrals

1. The following theorem says that a continuous function attains its average value:

   Theorem 1 (The Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that $f(c) = f_{\text{ave}}$. That is, there exists a $c \in [a, b]$ such that*

   $$\int_a^b f(x)\,dx = f(c)(b-a).$$

2. The following result says that if f is continuous and non-negative, then a vanishing integral implies the function vanishes:

   Corollary 1 (Vanishing Theorem)**.** *Suppose $f(x)$ is a continuous function on $[a, b]$. If $f(x) \geq 0$ for all $x \in [a, b]$ and*

   $$\int_a^b f(x)\,dx = 0,$$

   then $f(x) = 0$ for all $x \in [a, b]$.

3. We can "move" the average integral inside of a function that is convex:

   Theorem 2 (Jensen's Inequality)**.** *If $f(x)$ is a continuous function on $[a, b]$ and $g''(x) \geq 0$ for all x in the range of f, then $g(f_{\text{ave}}) \leq (g \circ f)_{\text{ave}}$. That is,*

   $$g\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b g(f(x))\,dx.$$

3.2 Example Problems

Problem 3.1. ($\star\star\star$) Prove Jensen's inequality. That is, suppose that $f(x)$ is a continuous function on $[a,b]$ and $g''(x) \geq 0$ for all x. Prove that

$$g\bigg(\frac{1}{b-a}\int_a^b f(x)\,dx\bigg) \leq \frac{1}{b-a}\int_a^b g(f(x))\,dx.$$

Solution 3.1. Let $L(x)$ be the linear approximation of g at the point $x = f_{\text{ave}}$. Since $g''(x) \geq 0$, we have $g(x)$ lies above its tangent line so,

$$L(x) := g(f_{\text{ave}}) + g'(f_{\text{ave}})(x - f_{\text{ave}}) \leq g(x).$$

Since this holds for all x, we have $L(f(x)) \leq g(f(x))$. Taking the average integral of both sides and using the monotonicity of definite integrals, we can conclude

$$g(f_{\text{ave}}) = \frac{1}{b-a}\int_a^b g(f_{\text{ave}}) + g'(f_{\text{ave}})(f(x) - f_{\text{ave}})\,dx \leq \frac{1}{b-a}\int_a^b g(f(x))\,dx.$$

Remark: We only require $g''(x) \geq 0$ on the range of f for the inequality to hold. This is because f_{ave} is in the range of f by the mean value theorem of integration, and g is only evaluated at points in the range of f. If $g''(x) > 0$, then the same proof shows $g(f_{\text{ave}}) < (g \circ f)_{\text{ave}}$ provided that f is not a constant function.

Remark: It is easy to see that the opposite inequality holds if $g''(x) \leq 0$. Suppose that $g''(x) \leq 0$, then $-g''(x) \geq 0$, so Jensen's inequality implies that

$$-g(f_{\text{ave}}) \leq (-g \circ f)_{\text{ave}} \implies g(f_{\text{ave}}) \geq (g \circ f)_{\text{ave}}.$$

Problem 3.2. ($\star\star$) Prove the midpoint Riemann sum is an under approximation of the definite integral if $f(x)$ is a convex function.

Solution 3.2. Recall that the midpoint Riemann sum is given by

$$\sum_{i=1}^n f\Big(a + \Big(i - \frac{1}{2}\Big)\Delta x\Big)\Delta x.$$

We need to show that

$$\sum_{i=1}^n f\Big(a + \Big(i - \frac{1}{2}\Big)\Delta x\Big)\Delta x \leq \int_a^b f(x)\,dx = \sum_{i=1}^n \int_{a+(i-1)\Delta x}^{a+i\Delta x} f(x)\,dx.$$

It suffices to show that the midpoint Riemann sum approximation using one rectangle is an under approximation of the definite integral,

$$(b-a)f\Big(\frac{a+b}{2}\Big) \leq \int_a^b f(x)\,dx.$$

The result is immediate from Jensen's inequality. If $f''(x) \geq 0$ on $[a,b]$, then

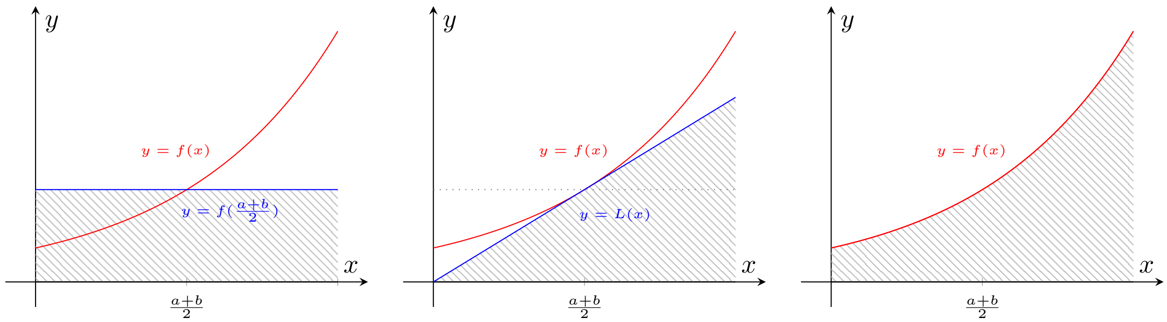
$$f\Big(\frac{a+b}{2}\Big) = f\Big(\frac{1}{b-a}\int_a^b x\,dx\Big) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \implies (b-a)f\Big(\frac{a+b}{2}\Big) \leq \int_a^b f(x)\,dx,$$

as required. This implies that

$$f\Big(a + \Big(i - \frac{1}{2}\Big)\Delta x\Big)\Delta x \leq \int_{a+(i-1)\Delta x}^{a+i\Delta x} f(x)\,dx \quad \text{for all } 1 \leq i \leq n,$$

so the midpoint Riemann sum is an under approximation when f is convex.

Remark. We can visualize this proof by noticing that the linear approximation $L(x)$ at the midpoint of the interval is an under approximation of a convex function. In the figure below, the area of the first two shaded regions are equal and smaller than the shaded region under the last graph. The shaded area of the left graph is the area of the midpoint Riemann sum, while the shaded area of the right graph is the area under the curve.



Problem 3.3. ($\star\star\star$) Prove the Mean Value Theorem for Integration by applying the Mean Value Theorem to the function $F(x) = \int_a^x f(t)\,dt$.

Solution 3.3. By the fundamental theorem of calculus, $F(x)$ is continuous on $[a,b]$ and $F(x)$ is differentiable on (a,b). Therefore, by the mean value theorem there exists a $c \in (a,b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c) = f(c).$$

Since $F(b) - F(a) = \int_a^b f(x)\,dx - \int_a^a f(x)\,dx = \int_a^b f(x)\,dx$, there exists a $c \in (a,b)$ such that

$$\frac{\int_a^b f(x)\,dx}{b-a} = f(c) \implies \int_a^b f(x)\,dx = f(c)(b-a).$$

Problem 3.4. ($\star\star\star$) Prove the vanishing theorem. That is, suppose that $f(x)$ is a continuous function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and

$$\int_a^b f(x)\,dx = 0.$$

Prove that $f(x) = 0$ for all $x \in [a,b]$.

Solution 3.4. On the contrary, suppose that $f(x^*) > 0$ for some point $x^* \in [a,b]$. Then by continuity, we must also have that $f(x) > 0$ on some interval $[k,\ell] \subset [a,b]$. By the mean value theorem of integration, there exists a $c \in [k,\ell]$ such that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k).$$

Since we also have that $f(x) > 0$ for all $x \in [k,\ell]$, we must have $f(c) > 0$, which implies that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k) > 0.$$

Since $f(x) \geq 0$, monotonicity of integration and the conclusion above implies

$$\int_a^b f(x)\,dx \geq \int_k^\ell f(x)\,dx > 0,$$

which contradicts the fact that $\int_a^b f(x)\,dx = 0$. Therefore, we must have that $f(x) = 0$ for all $x \in [a,b]$.