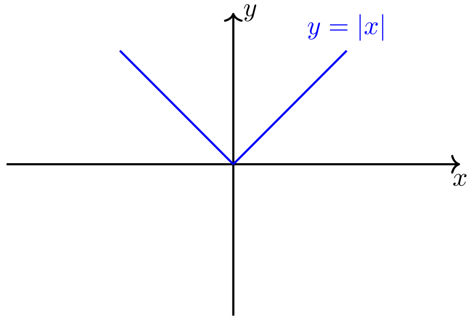
1 Absolute Value

For $x \in \mathbb{R}$, the absolute value of x is a piecewise function defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

The graph is displayed below:



Basic Properties: The absolute value function satisfies the following properties

1. Non-negativity: $|x| \geq 0$
2. Multiplicativity: $|xy| = |x||y|$
3. Positive Definiteness: $|x| = 0$ if and only if $x = 0$
4. Triangle Inequality: $|x + y| \leq |x| + |y|$
5. Reverse Triangle Inequality: $||x| - |y|| \leq |x - y|$.

1.1 Example Problems

1.1.1 Absolute Value Inequalities:

Strategy: We can proceed in two ways:

1. In general, we need to break our region into cases where our absolute values change sign and solve the inequality on each region separately.
2. Shortcut: If we want to compute $|f(x)| \leq g(x)$ or $|f(x)| \geq g(x)$ then we can replace the absolute value with $\pm$ and solve the two cases corresponding to $+f(x)$ and $-f(x)$.

Problem 1.1. ($\star$) Find all x such that

$$|2x - 4| \leq |x + 3|.$$

Solution 1.1. Rearranging the inequality, we see that

$$|2x - 4| \leq |x + 3| \Rightarrow \left|\frac{2x - 4}{x + 3}\right| \leq 1 \Rightarrow \pm\frac{2x - 4}{x + 3} \leq 1.$$

Solving for the case with the positive sign, implies

$$\frac{2x - 4}{x + 3} \leq 1 \Rightarrow 2x - 4 \leq x + 3 \Rightarrow x \leq 7$$

and for the case with the negative sign, implies

$$-\frac{2x-4}{x+3} \le 1 \Rightarrow -2x+4 \le x+3 \Rightarrow -3x \le -1 \Rightarrow x \ge \frac{1}{3}.$$

Therefore, the inequality is satisfied for

$$\frac{1}{3} \le x \le 7.$$

Remark: Since $x = -3$ is not a solution, we do not run into the issue of dividing by zero.

Problem 1.2. ($\star$) Find all x such that

$$|x+8| < 5x+10.$$

Solution 1.2. The function $|x+8|$ changes sign when $x = -8$, so we consider the regions $x < -8$ and $x > -8$.

1. $x > -8$: In this case we have $|x+8| = x+8$, so solving the inequality gives
$$|x+8| < 5x+10 \Rightarrow x+8 < 5x+10 \Rightarrow x > -\frac{1}{2}.$$
Since we must have both $x \ge -8$ and $x > -\frac{1}{2}$, we have our inequality is satisfied when $x > -\frac{1}{2}$.
2. $x < -8$: In this case we have $|x+8| = -(x+8)$ so solving the inequality gives
$$|x+8| < 5x+10 \Rightarrow -x-8 < 5x+10 \Rightarrow x > -3.$$
Since we must have both $x < -8$ and $x > -3$, no x in this region satisfies our inequality.
3. $x = 8$: When $x = -8$, $0 < -40+10$ is a false statement, so $x = -8$ does not satisfy our inequality.

Combining the cases above, our solutions are $x > -\frac{1}{2}$.

Remark: We can also solve $\pm(x+8) < 5x+10$ like in Problem 1.1 to conclude that $x > -\frac{1}{2}$.

Problem 1.3. ($\star\star$) Find all x such that

$$|x-2| < |x+4| - 2.$$

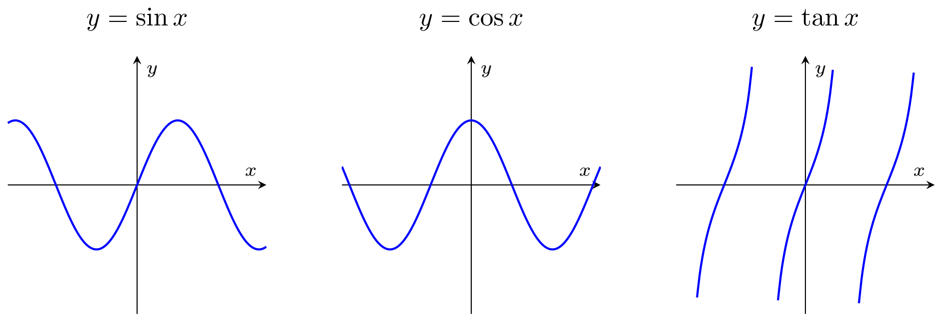
Solution 1.3. The function $|x-2|$ changes sign when $x = 2$ and $|x+4|$ changes sign when $x = -4$, so we consider the cases

1. $x < -4$: On this region, we have $|x-2| = -x+2$ and $|x+4| = -x-4$ so we have
$$|x-2| < |x+4| - 2 \Rightarrow -x+2 < -x-4-2 \Rightarrow 8 < 0,$$
which is a false expression, so no x in this region satisfies our inequality.
2. $-4 < x < 2$: On this region, we have $|x-2| = -x+2$ and $|x+4| = x+4$ so we have
$$|x-2| < |x+4| - 2 \Rightarrow -x+2 < x+4-2 \Rightarrow x > 0.$$
so we must have $x > 0$ and $-4 < x < 2$ which means $0 < x < 2$ is a solution to the inequality.
3. $x > 2$: On this region, we have $|x-2| = x-2$ and $|x+4| = x+4$ so we have
$$|x-2| < |x+4| - 2 \Rightarrow x-2 < x+4-2 \Rightarrow 0 < 4,$$
which is a true expression so $x > 2$ is a solution to the inequality.
4. $x = -4$ or $x = 2$: When $x = -4$, we have $6 < -2$ which is false, so $x = -4$ is not a solution. When $x = 2$, we have $0 < 4$ which is true, so $x = 2$ is a solution.

Combining our cases above, our solutions are $x > 0$.

2 Trigonometric Functions

The 3 main trigonometric functions discussed in this course are $\sin(x)$, $\cos(x)$ and $\tan(x)$:



Key Values: The key values of $\sin(x)$ and $\cos(x)$ can be summarized by the table of values

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

The other key values can be extrapolated by looking at the shapes of the graphs.

Basic Properties: The key trigonometric identities are

1. Pythagorean Identity:
$$\sin^2(\theta) + \cos^2(\theta) = 1.$$
2. Sum and Difference Formulas:
$$\sin(\theta \pm \varphi) = \sin(\theta)\cos(\varphi) \pm \cos(\theta)\sin(\varphi), \qquad \cos(\theta \pm \varphi) = \cos(\theta)\cos(\varphi) \mp \sin(\theta)\sin(\varphi).$$

From these, one can derive the following identities

3. Symmetry and Periodicity: (Use the Sum and Difference Formulas)
$$\sin(-\theta) = -\sin(\theta), \quad \cos(-\theta) = \cos(\theta), \quad \sin(\theta + 2k\pi) = \sin(\theta), \quad \cos(\theta + 2k\pi) = \cos(\theta)$$
4. Complementary and Supplementary Angles: (Use the Sum and Difference Formulas)
$$\sin\Big(\frac{\pi}{2} - \theta\Big) = \cos(\theta), \quad \sin(\pi - \theta) = \sin(\theta), \quad \cos\Big(\frac{\pi}{2} - \theta\Big) = \sin(\theta), \quad \cos(\pi - \theta) = -\cos(\theta).$$
5. Double Angle Formulas: (Use the Sum and Difference Formulas and the Pythagorean Identity)
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) = 2\cos^2(\theta) - 1.$$
6. Half Angle Formulas: (Use the Double Angle Formulas for $\cos(2\theta)$)
$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}, \qquad \cos^2(\theta) = \frac{1 + \cos 2\theta}{2}.$$
7. Product to Sum Formulas: (Use the Sum and Difference Formulas)
$$\cos(\theta)\cos(\varphi) = \frac{1}{2}(\cos(\theta + \varphi) + \cos(\theta - \varphi)), \quad \sin(\theta)\sin(\varphi) = \frac{1}{2}(\cos(\theta - \varphi) - \cos(\theta + \varphi)),$$
$$\sin(\theta)\cos(\varphi) = \frac{1}{2}(\sin(\theta + \varphi) + \sin(\theta - \varphi)).$$

To solve some word problems, it is also useful to recall the *Cosine Law*:
$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{where } C \text{ is the angle opposite side } c.$$

2.1 Example Problems

2.1.1 General Trigonometry Problems

Problem 2.1. ($\star$) Find the value of

$$\sin\Big(-\frac{3}{4}\pi\Big).$$

Solution 2.1. We will reduce the problem to one of the key values for sine or cosine. Since $\sin(x)$ is odd,

$$\sin\Big(-\frac{3}{4}\pi\Big) = -\sin\Big(\frac{3}{4}\pi\Big).$$

Using the supplementary angle identity,

$$\sin(\theta) = \sin(\pi - \theta) \implies -\sin\Big(\frac{3}{4}\pi\Big) = -\sin\Big(\pi - \frac{3}{4}\pi\Big) = -\sin\Big(\frac{1}{4}\pi\Big) = -\frac{\sqrt{2}}{2}.$$

Problem 2.2. ($\star$) Find all x such that

$$\cos\Big(x + \frac{\pi}{2}\Big) = 0.$$

Solution 2.2. From the graph of $\cos(x)$, we know $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$. Therefore, the solutions of our equation are x such that

$$x + \frac{\pi}{2} = \frac{\pi}{2} + k\pi \Rightarrow x = k\pi \text{ for } k \in \mathbb{Z}.$$

Problem 2.3. ($\star\star$) Let $x \in [0, 2\pi)$. How many solutions does

$$(2\cos(x) - 1)(\cos^2(x) - 1)(\sin(x) + 5)(\cos(x + 5) - 1) = 0$$

have?

Solution 2.3. We have $(2\cos(x) - 1)(\cos^2(x) - 1)(\sin(x) + 5)(\cos(x + 5) - 1) = 0$ if and only if

$$2\cos(x) - 1 = 0 \text{ or } \cos^2(x) - 1 = 0 \text{ or } \sin(x) + 5 = 0 \text{ or } \cos(x + 5) - 1 = 0.$$

Notice that

1. $2\cos(x) - 1 = 0 \Rightarrow \cos(x) = \frac{1}{2}$ has 2 solutions in the interval $[0, 2\pi)$.
2. $\cos^2(x) - 1 = 0 \Rightarrow \cos^2 x = 1 \Rightarrow \cos(x) = \pm 1$ has 2 solutions in the interval $[0, 2\pi)$.
3. $\sin(x) + 5 = 0$ has no solutions since the range of $\sin(x)$ is $[-1, 1]$.
4. $\cos(x+5) - 1 = 0 \Rightarrow \cos(x+5) = 1 \Rightarrow x + 5 = 2k\pi \Rightarrow x = 2k\pi - 5$ has 1 solutions in the interval $[0, 2\pi)$ when $k = 1$.

None of the solutions coincide, so there are 5 solutions in total.

Problem 2.4. ($\star\star\star$) Derive the Sum and Difference formulas:

$$\sin(\theta \pm \varphi) = \sin(\theta)\cos(\varphi) \pm \cos(\theta)\sin(\varphi), \qquad \cos(\theta \pm \varphi) = \cos(\theta)\cos(\varphi) \mp \sin(\theta)\sin(\varphi).$$

Solution 2.4. Recall Euler's Identity,

$$e^{ix} = \cos(x) + i\sin(x).$$

If we take $x = \theta + \varphi$, then

$$e^{i(\theta+\varphi)} = \cos(\theta+\varphi) + i\sin(\theta+\varphi)$$

and using the fact $\exp(a+b) = \exp(a)\exp(b)$, we also have

$$\begin{aligned} e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi} &= (\cos(\theta) + i\sin(\theta))(\cos(\varphi) + i\sin(\varphi)) \\ &= \Big(\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)\Big) + i\Big(\sin(\theta)\cos(\varphi) + \sin(\varphi)\cos(\theta)\Big). \end{aligned}$$

Therefore, our two equations above implies

$$\cos(\theta+\varphi) + i\sin(\theta+\varphi) = e^{i(\theta+\varphi)} = \Big(\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)\Big) + i\Big(\sin(\theta)\cos(\varphi) + \sin(\varphi)\cos(\theta)\Big).$$

Equating the real and imaginary parts, we have

$$\cos(\theta+\varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) \text{ and } \sin(\theta+\varphi) = \sin(\theta)\cos(\varphi) + \sin(\varphi)\cos(\theta).$$

To derive the formulas for $\theta - \varphi$, we use the fact $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ and use our formulas above to conclude

$$\cos(\theta-\varphi) = \cos(\theta+(-\varphi)) = \cos(\theta)\cos(-\varphi) - \sin(\theta)\sin(-\varphi) = \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)$$

and

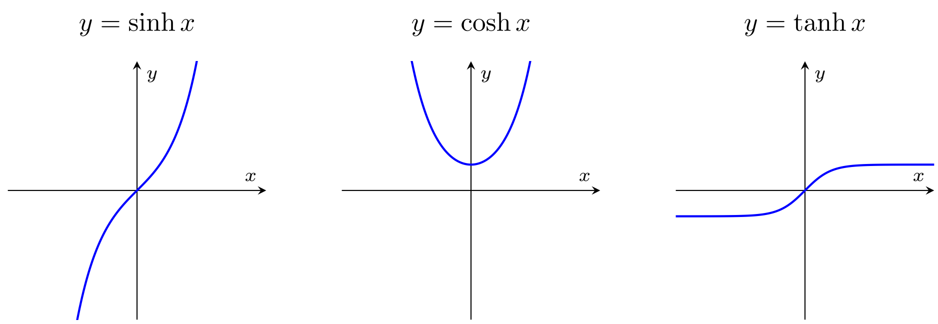
$$\sin(\theta-\varphi) = \sin(\theta+(-\varphi)) = \sin(\theta)\cos(-\varphi) + \sin(-\varphi)\cos(\theta) = \sin(\theta)\cos(\varphi) - \sin(\varphi)\cos(\theta).$$

3 Hyperbolic Functions

The 3 main hyperbolic functions discussed in this course are

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

The graphs of these functions are displayed below:



Basic Properties: Just like the trigonometric functions, the hyperbolic functions satisfy a similar set of identities.

1. Analogue of the "Pythagorean" Identity:

$$\cosh^2(x) - \sinh^2(x) = 1.$$

2. Sum and Difference Formulas:

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \sinh(x)\cosh(y), \qquad \cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y).$$

3. Double Angle Formulas: (Use sum and difference formulas and the Pythagorean identity)

$$\sinh(2x) = 2\sinh(x)\cosh(x), \qquad \cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\sinh^2(x) + 1 = 2\cosh^2(x) - 1.$$

4. Half Angle Formulas: (Use the double angle formula for $\cosh(2x)$)

$$\sinh^2(x) = \frac{\cosh(2x) - 1}{2}, \qquad \cosh^2(x) = \frac{\cosh(2x) + 1}{2}.$$

3.1 Example Problems

Problem 3.1. ($\star\star$) Verify the Pythagorean identity

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Solution 3.1. This is a direct computation. We have

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^y + e^{-y}}{2}\right)^2 + \left(\frac{e^y - e^{-y}}{2}\right)^2 = \frac{e^{2y} + 2 + e^{-2y} - e^{2y} + 2 - e^{-2y}}{4} = 1.$$