1 Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation involving a function $y(x)$ and its derivatives,

$$F(x, y, y', y'', \dots, y^{(n)}) = 0.$$

The order of the differential equation is the highest order derivative appearing in the equation. Our goal is to find an explicit formula for the function $y(x)$ that satisfies the differential equation.

1.1 Slope Fields

The solution to a first order differential equation can be visualized using a slope field. Any solution $y(x)$ to the ODE

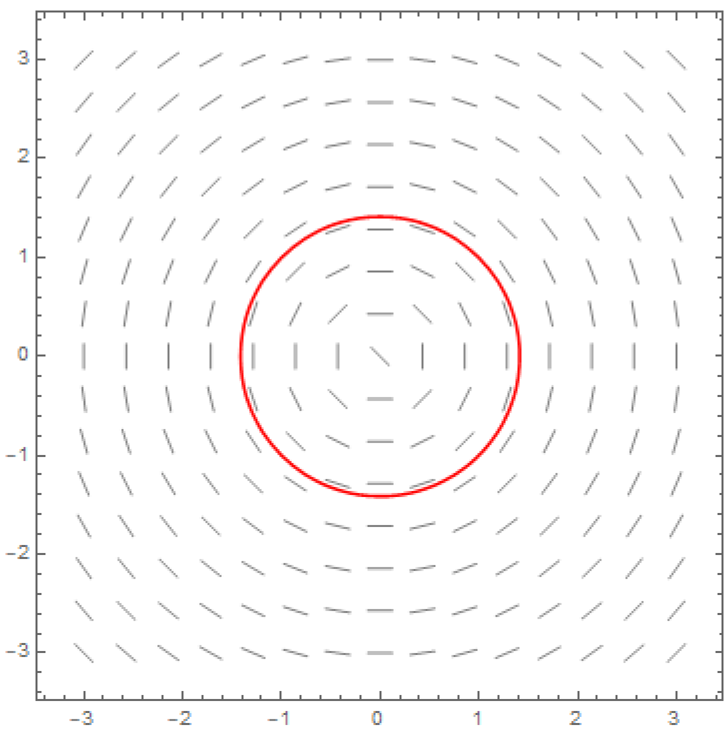
$$\frac{dy}{dx} = f(x, y)$$

must have slope $y'(x_0) = f(x_0, y_0)$ at the point $(x_0, y(x_0))$. The slope field is a plot in $\mathbb{R}^2$ such that every point $(x_0, y(x_0))$ corresponds to a line segment with slope $f(x_0, y_0)$. The solution $y(x)$ follow the directions of the slope field, so we can use the slope fields to visualize the family of solutions to differential equations.

Example 1. The slope field for the ODE

$$\frac{dy}{dx} = -\frac{x}{y}$$

is displayed below



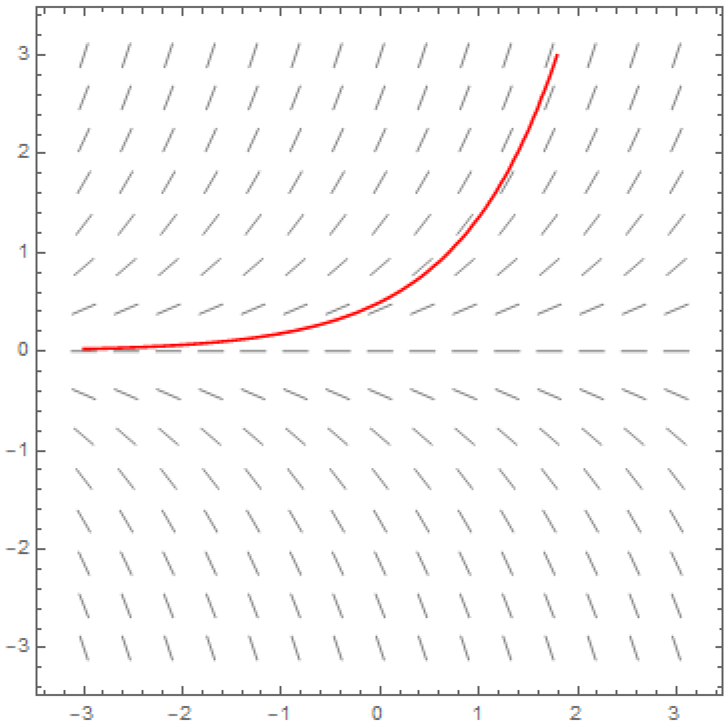
It looks like solutions to this differential equation are circles. A particular solution is displayed in red. In the next section we will use integration to show that $x^2 + y^2 = C$ are solutions to this differential equation. We can check that this is a solution by implicitly differentiating both sides,

$$x^2 + y^2 = C \implies 2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Example 2. The slope field for the ODE

$$\frac{dy}{dx} = y$$

is displayed below



It looks like the solutions to this differential equation grow exponentially. A particular solution is displayed in red. In the next section we will use integration to show that $y = Ce^x$ are solutions to this differential equation. We can check that this is a solution by differentiating both sides,

$$y = Ce^x \implies \frac{dy}{dx} = Ce^x \implies \frac{dy}{dx} = y \qquad \text{since } y = Ce^x.$$

1.2 Separable Differential Equations

Suppose that we want to find a function y such that it satisfies a differential equation of the form

$$\frac{dy}{dx} = f(x)g(y)$$

for some functions $f(x)$ and $g(y)$. We can find the solution by separating variables and integrating both sides,

$$\frac{dy}{dx} = f(x)g(y) \implies \frac{dy}{g(y)} = f(x)dx \implies \int \frac{1}{g(y)}\,dy = \int f(x)\,dx \implies G(y) = F(x) + C$$

where $G(y)$ is an antiderivative of $\frac{1}{g(y)}$ and $F(x)$ is an antiderivative of $f(x)$. This procedure gives an implicit formula formula for a function y that satisfies the differential equation. If we are given an initial condition, then we can solve for the integration constant C.

1.3 Example Problems

Problem 1. ($\star$) The rate of growth of a population P is modeled by

$$\frac{dP}{dt} = kP$$

where $k \neq 0$. Suppose that the initial population $P(0) = P_0$ for some constant $P_0 > 0$. Find the population function $P(t)$. How long will it take for the population to double?

Solution 1. Separating variables and integrating, we have

$$\frac{dP}{dt} = kP \Rightarrow \int \frac{dP}{P} = \int k dt \Rightarrow \ln|P| = kt + C$$

Since $P(t)$ must be positive in a reasonable model, $|P| = P$, so we can exponentiate both sides to conclude

$$P = e^{kt+C} = e^C e^{kt}.$$

To solve for the integrating constants C, since $P(0) = P_0$, we have

$$P_0 = P(0) = e^C e^{k\cdot 0} = e^C \Rightarrow \ln P_0 = C.$$

Therefore, the population is given by

$$P(t) = e^{\ln P_0} e^{kt} = P_0 e^{kt}.$$

To find the time for the population to double, we want to find the t such that $P(t) = 2P_0$. That is,

$$2P_0 = P_0 e^{kt} \Rightarrow e^{kt} = 2 \Rightarrow kt = \ln(2) \Rightarrow t = \frac{\ln(2)}{k}.$$

In particular, if $k < 0$ (we have a decreasing population) then we have our population will never double.

Problem 2. ($\star\star$) A pizza is put in a 200°C oven and heats up according to the differential equation

$$\frac{dH}{dt} = -k(H - 200), \text{ where } k > 0.$$

The pizza is put in the oven at 20°C and is removed 30 minutes later at a temperature of 120°C. Find the proportionality constant k.

Solution 2. The general solution to the ODE

$$\frac{dH}{dt} = -k(H - 200),$$

can be solved using separation of variables,

$$\frac{dH}{dt} = -k(H - 200) \Rightarrow \frac{dH}{(H - 200)} = -kt\, dt \Rightarrow \ln|H - 200| = -kt + C$$

Solving for H and using the fact $H - 200 < 0$ in a reasonable model, we see that

$$\ln|H - 200| = -kt + C \Rightarrow \ln(200 - H) = -kt + C \Rightarrow H = 200 - De^{-kt}$$

where $D = e^C > 0$. To find D, we can use the fact that at $H(0) = 20$,

$$20 = H(0) = 200 - D \implies D = 180.$$

Since $H(30) = 120$, we have

$$120 = H(30) = 200 - 180 \cdot e^{-30k} \implies k = -\frac{1}{30} \cdot \ln \frac{80}{180} \approx 0.027.$$

Problem 3. ($\star$) Suppose the velocity of a particle is given by

$$v(t) := \frac{ds}{dt} = \sin(t) - \cos(t).$$

Find the position function $s(t)$ of the particle given that $s(0) = 0$.

Solution 3. The rate of change of position is velocity, so we have the differential equation

$$\frac{ds}{dt} = \sin(t) - \cos(t).$$

Separating variables and integrating, we have

$$\frac{ds}{dt} = \sin(t) - \cos(t) \Rightarrow \int ds = \int \sin(t) - \cos(t)\, dt \Rightarrow s = -\cos(t) - \sin(t) + C.$$

To solve for the integrating constant, since $s(0) = 0$, we have

$$0 = s(0) = -\cos(0) - \sin(0) + C = -1 + C \Rightarrow C = 1.$$

Therefore, the position is given by

$$s(t) = -\cos(t) - \sin(t) + 1.$$

Problem 4. ($\star\star$) Suppose the acceleration of a particle is given by

$$a(t) := \frac{d^2 s}{dt^2} = t + 1.$$

Find the position function $s(t)$ of the particle given that $s(0) = 0$ and $s(1) = 2$.

Solution 4. The rate of change of velocity is acceleration, so we have the differential equation

$$\frac{dv}{dt} = t + 1.$$

Separating variables and integrating, we have

$$\frac{dv}{dt} = t + 1 \Rightarrow \int dv = \int t + 1\, dt \Rightarrow v = \frac{t^2}{2} + t + C.$$

The rate of change of position is velocity, so using the fact above, we have the differential equation

$$\frac{ds}{dt} = \frac{t^2}{2} + t + C.$$

Separating variables and integrating, we have

$$\frac{ds}{dt} = \frac{t^2}{2} + t + C \Rightarrow \int ds = \int \frac{t^2}{2} + t + C\, dt \Rightarrow s = \frac{t^3}{6} + \frac{t^2}{2} + Ct + D.$$

To solve for the integrating constants C and D, since $s(0) = 0$, we have

$$0 = s(0) = D \Rightarrow D = 0.$$

And since $s(1) = 2$, we have

$$2 = s(1) = \frac{1}{6} + \frac{1}{2} + C \Rightarrow C = 2 - \frac{1}{6} - \frac{1}{2} = \frac{4}{3}$$

Therefore, the position is given by

$$s(t) = \frac{t^3}{6} + \frac{t^2}{2} + \frac{4t}{3}.$$

Problem 5. ($\star\star$) Solve

$$\frac{dy}{dx} = yx.$$

Solution 5.

Separate Variables: Separating variables and integrating, we have

$$\frac{dy}{dx} = yx \Rightarrow \int \frac{1}{y}\, dy = \int x\, dx \Rightarrow \ln|y| = \frac{x^2}{2} + A,$$

for some constant A. Solving for y we get

$$|y| = e^{\frac{x^2}{2}+A} \Rightarrow y = \pm e^A e^{\frac{x^2}{2}}.$$

If we define the non-zero constant $B = \pm e^A$, then we have

$$y = Be^{\frac{x^2}{2}}, \text{ where } B \text{ is some non-zero constant.}$$

However, notice that $y \equiv 0$ also satisfies $\frac{dy}{dx} = yx$, so $y = 0$ is also a solution. Therefore, the most general form of our solution is

$$y = Ce^{\frac{x^2}{2}}, \text{ where } C \text{ is some constant.}$$

Remark: We can check that $y(x) = Ce^{\frac{x^2}{2}}$ satisfies our differential equation. Notice that by the chain rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}Ce^{\frac{x^2}{2}} = \underbrace{Ce^{\frac{x^2}{2}}}_{y} \cdot x = yx.$$

Remark: This example also explains how to remove the absolute value sign that appears when we take the antiderivative of $\frac{1}{y}$ and the usual approach one can take to absorb the resulting plus or minus sign into the constant of integration. For example, the solution for the population growth model in Problem 1 can be extended to negative populations or 0 initial populations using this argument.

Problem 6. ($\star$) Solve

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = 4.$$

Solution 6.

Separate Variables: Separating variables and integrating, we have

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \int y\, dy = -\int x\, dx \Rightarrow y^2 = -x^2 + C,$$

for some constant C.

Solving for C: Using the initial condition $y(0) = 4$, we must have

$$4^2 = -0^2 + C \Rightarrow C = 16.$$

Therefore, the implicit particular solution to this ODE is

$$y^2 + x^2 = 16.$$

Remark: If we want to write our solution as a function $y(x)$, then

$$y = \sqrt{16 - x^2}.$$

We chose the positive square root, because we need the point $(0, y(0)) = (0, 4)$ to lie on the curve.

Problem 7. ($\star$) Solve

$$\frac{dy}{dx} = \frac{1}{x^2} \cdot \left(\frac{1}{y^2} - \frac{2}{y^3}\right)^{-1}, \qquad y(1) = 1.$$

Solution 7.

Separate Variables: Separating variables and integrating, we have

$$\frac{dy}{dx} = \frac{1}{x^2} \cdot \left(\frac{1}{y^2} - \frac{2}{y^3}\right)^{-1} \Rightarrow \left(\frac{1}{y^2} - \frac{2}{y^3}\right) dy = \frac{1}{x^2}\, dx \Rightarrow -y^{-1} + y^{-2} = -x^{-1} + C$$

for some constant C.

Solving for C: Using the initial condition $y(1) = 1$, we must have

$$-1^{-1} + 1^{-2} = -1^{-1} + C \Rightarrow C = 1.$$

Therefore, the implicit particular solution to this ODE is

$$-y^{-1} + y^{-2} + x^{-1} = 1.$$

Problem 8. ($\star\star\star$) Justify the technique used to solve separable ordinary differential equations:

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\, dx \implies G(y) = F(x) + C$$

where $G(y)$ is an antiderivative of $\frac{1}{g(y)}$ and $F(x)$ is an antiderivative of $f(x)$.

Solution 8. Using the notation $y'(x) = \frac{dy}{dx}$ and writing $y = y(x)$ explicitly as a function of x, we have

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{y'(x)}{g(y(x))} = f(x) \Rightarrow \int \frac{y'(x)}{g(y(x))}\, dx = \int f(x)\, dx.$$

Using the change of variables $u = y(x)$ on the first integral involving the $y(x)$ term, we see

$$\int \frac{y'(x)}{g(y(x))}\, dx = \int \frac{du}{g(u)} = \int \frac{dy}{g(y)}.$$

Therefore, using this change of variables, we can conclude that

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\, dx.$$

This means there is a hidden change of variables that goes on when we formally separated $\frac{dy}{dx}$ in the second step of the technique.