1 Applications in Probability

In this section, we introduce an application of single variable integration in probability.

1.1 Continuous Random Variables

A *random variable* is a variable that takes values depending on the outcome of a random event. More precisely, a random variable is a function from a space of outcomes to $\mathbb{R}$. The likelihood that a random variable takes a certain range of values is encoded by a density function:

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is called a *probability density function* (p.d.f.) if

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$; and
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Definition 2. A random variable X is *continuous* if there exists a density function f such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

That is, the probability that random variable takes values in $[a, b]$ is given by integrating the density of X from a to b. Since f is positive and integrates to 1, all probabilities take values in $[0, 1]$.

Definition 3. The *cumulative density function* (c.d.f.) of X is a function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f(x)\,dx.$$

We can recover the p.d.f. by differentiating the cumulative density function.

Definition 4. The *expected value* or *mean* is the weighted average value of the random variable,

$$\mathbb{E}\,X = \int_{-\infty}^{\infty} x f(x)\,dx.$$

In general, if $g : \mathbb{R} \to \mathbb{R}$, then the expected value of $g(X)$ is given by

$$\mathbb{E}\,g(X) = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

Definition 5. The *variance* is how far the random variable usually deviates from its mean,

$$\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}\,X)^2 = \mathbb{E}\,X^2 - (\mathbb{E}\,X)^2.$$

If $\mathbb{E}\,X$ does not exist or is equal to $\pm\infty$, then $\mathrm{Var}(X)$ is undefined.

1.2 Important Density Functions

1. Uniform Distribution: Let $L < R$. The notation $X \sim \mathrm{Uniform}[L, R]$ means X has density
$$f(x) = \frac{1}{R - L} \mathbb{1}(L \leq x \leq R) = \begin{cases} \frac{1}{R-L} & x \in [L, R] \\ 0 & \text{otherwise.} \end{cases}$$
2. Exponential Distribution: Let $\lambda > 0$. The notation $X \sim \mathrm{Exponential}(\lambda)$ means X has density
$$f(x) = \lambda e^{-\lambda x} \mathbb{1}(x \geq 0) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$
3. Normal Distribution: Let $\mu \in \mathbb{R}$, $\sigma > 0$. The notation $X \sim \mathrm{N}(\mu, \sigma^2)$ means X has density
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

1.3 Example Problems

Problem 1. ($\star$) Suppose $X \sim \text{Uniform}[L, R]$. Find $\mathbb{E}\, X$ and $\text{Var}(X)$.

Solution 1.

(a) We have

$$\mathbb{E}\, X = \frac{1}{R-L}\int_L^R x\,dx = \frac{1}{R-L}\frac{x^2}{2}\bigg|_{x=L}^{x=R} = \frac{R+L}{2}.$$

(b) Since

$$\mathbb{E}\, X^2 = \frac{1}{R-L}\int_L^R x^2\,dx = \frac{1}{R-L}\frac{x^3}{3}\bigg|_{x=L}^{x=R} = \frac{R^3-L^3}{3(R-L)} = \frac{L^2+LR+R^2}{3},$$

we have

$$\text{Var}(X) = \mathbb{E}\, X^2 - (\mathbb{E}\, X)^2 = \frac{L^2+LR+R^2}{3} - \Big(\frac{R+L}{2}\Big)^2 = \frac{1}{12}(R-L)^2.$$

Problem 2. ($\star$) Suppose $X \sim \text{Uniform}[a, b]$ and f is a continuous function defined on $[a, b]$. Find $\mathbb{E}\, f(X)$.

Solution 2. By definition,

$$\mathbb{E}\, f(X) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This is precisely the average value of a function f_{ave} that we defined in Week 2.

Problem 3. ($\star$) Suppose $X \sim \text{Exponential}(\lambda)$. Find F_X, $\mathbb{E}\, X$ and $\text{Var}(X)$.

Solution 3.

(a) There are two cases. For $t \geq 0$, we have

$$F_X(t) = \int_0^t \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\bigg|_{x=0}^{x=t} = 1 - e^{-\lambda t}.$$

For $t < 0$, we have the trivial case

$$F_X(t) = \int_{-\infty}^t f(x)\,dx = \int_{-\infty}^t 0\,dx = 0.$$

(b) Using integration by parts, we have

$$\mathbb{E}\, X = \int_0^\infty \lambda x e^{-\lambda x}\,dx = -xe^{-\lambda x} - \frac{1}{\lambda}e^{-\lambda x}\bigg|_{x=0}^{x=\infty} = \lim_{t\to\infty} -te^{-\lambda t} - \frac{1}{\lambda}e^{-\lambda t} + \frac{1}{\lambda} = \frac{1}{\lambda}.$$

(c) Using integration by parts, we have

$$\begin{aligned}\mathbb{E}\, X^2 = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx &= -x^2e^{-\lambda x} - \frac{2x}{\lambda}e^{-\lambda x} - \frac{2}{\lambda^2}e^{-\lambda x}\bigg|_{x=0}^{x=\infty} \\ &= \lim_{t\to\infty} -t^2e^{-\lambda t} - \frac{2t}{\lambda}e^{-\lambda t} - \frac{2}{\lambda^2}e^{-\lambda t} + \frac{2}{\lambda^2} \\ &= \frac{2}{\lambda^2}.\end{aligned}$$

Therefore,

$$\text{Var}(X) = \mathbb{E}\, X^2 - (\mathbb{E}\, X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Problem 4. ($\star\star$) Suppose $X \sim N(\mu, \sigma^2)$. Find $\mathbb{E}\,X$ and $\operatorname{Var}(X)$.

Solution 4. A common trick to compute integrals is to write the integral in terms of its density functions. We will use the fact that $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is a density function. That is, for all μ and $\sigma > 0$,

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

multiple times in the computations below. When working with $N(\mu,\sigma^2)$, it is usually a good idea to do the change of variables $u = \frac{x-\mu}{\sigma} \leftrightarrow x = \sigma u + \mu$ to reduce the problem to one involving the density of $N(0,1)$. This is called *standardization*.

(a) Using a change of variables $u = \frac{x-\mu}{\sigma} \leftrightarrow x = \sigma u + \mu$, we have

$$\begin{aligned}
\mathbb{E}\,X &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} xe^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma u + \mu)e^{-\frac{u^2}{2}}\,du \\
&= \frac{\sigma}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} ue^{-\frac{u^2}{2}}\,du}_{=0} + \mu\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du}_{=1} \\
&= \mu.
\end{aligned}$$

We used the fact that $ue^{-\frac{u^2}{2\sigma^2}}$ is odd and integrable, so $\int_{-\infty}^{\infty} ue^{-\frac{u^2}{2}}\,du = 0$ by symmetry.

(b) Using a change of variables $u = \frac{x-\mu}{\sigma} \leftrightarrow x = \sigma u + \mu$, we have

$$\begin{aligned}
\mathbb{E}\,X^2 &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^2e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma u + \mu)^2e^{-\frac{u^2}{2}}\,du \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^2e^{-\frac{u^2}{2}}\,du + \frac{2\sigma\mu}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} ue^{-\frac{u^2}{2}}\,du}_{=0} + \mu^2\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du}_{=1} \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^2e^{-\frac{u^2}{2}}\,du + \mu^2 && ue^{-u^2} \text{ is odd} \\
&= \underbrace{-\frac{\sigma^4}{\sqrt{2\pi}}ue^{-\frac{u^2}{2}}\Big|_{u=-\infty}^{u=\infty}}_{=0} + \sigma^2\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du}_{=1} + \mu^2 && \text{by parts} \\
&= \sigma^2 + \mu^2.
\end{aligned}$$

Therefore,

$$\operatorname{Var}(X) = \mathbb{E}\,X^2 - (\mathbb{E}\,X)^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

Problem 5. ($\star\star$) (Memoryless Property) Recall that the probability of A given B is defined by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

If $X \sim \text{Exponential}(\lambda)$, show that

$$\mathbb{P}(X \geq t + s \mid X \geq t) = \mathbb{P}(X \geq s)$$

for all $s, t \geq 0$.

Solution 5. If X is a continuous random variable, then

$$\mathbb{P}(X \geq t) = \int_t^\infty f(x)\,dx = \int_{-\infty}^\infty f(x)\,dx - \int_{-\infty}^t f(x)\,dx = 1 - \mathbb{P}(X \leq t) = 1 - F_X(t).$$

Therefore, if $X \sim \text{Exponential}(\lambda)$, the result in Problem 2 implies that

$$\mathbb{P}(X \geq t) = 1 - F_X(t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}.$$

Of course, we could have also computed $\mathbb{P}(X \geq t) = \int_t^\infty \lambda e^{-\lambda x}\,dx$ explicitly as well. Therefore,

$$\mathbb{P}(X \geq t + s \mid X \geq t) = \frac{\mathbb{P}(X \geq t + s \text{ and } X \geq s)}{\mathbb{P}(X \geq t)} = \frac{\mathbb{P}(X \geq t + s)}{\mathbb{P}(X \geq t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(X \geq s).$$

Problem 6. ($\star\star$) Suppose $X \sim N(0,1)$. For all $t \in \mathbb{R}$, show that

$$m_X(t) := \mathbb{E}\, e^{tX} = e^{\frac{t^2}{2}}.$$

The function $m_X(t) = e^{tX}$ is called the moment generating function of the standard Normal distribution.

Solution 6. By definition,

$$\mathbb{E}\, e^{tX} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{tx} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2 - 2tx}{2}}\,dx.$$

Completing the square in x numerator, we see that

$$x^2 - 2tx = x^2 - 2tx + t^2 - t^2 = (x - t)^2 - t^2.$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2 - 2tx}{2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-t)^2 - t^2}{2}}\,dx = e^{\frac{t^2}{2}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-t)^2}{2}}\,dx}_{=1} = e^{\frac{t^2}{2}}$$

since the $\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t^2)}{2}}$ is the density of a Gaussian random variable with mean t and variance 1.

Problem 7. ($\star\star\star$) Suppose $X \sim N(\mu, \sigma^2)$. For all $t \in \mathbb{R}$, show that

$$m_X(t) := \mathbb{E}\, e^{tX} = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

The function $m_X(t) = e^{tX}$ is called the moment generating function of the Normal distribution.

Solution 7. We will standardize the random variable to reduce it to the case in Problem 5. Using the change of variables $u = \frac{x-\mu}{\sigma} \Leftrightarrow x = \sigma u + \mu$, we see that

$$\mathbb{E}\, e^{tX} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty e^{t(\sigma u + \mu)} e^{-\frac{u^2}{2}}\,dx = e^{t\mu} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{t\sigma u} e^{-\frac{u^2}{2}}\,du.$$

From Problem 5, we know that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{tx} e^{-\frac{x^2}{2}}\,dx = e^{\frac{t^2}{2}} \implies \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{(t\sigma)u} e^{-\frac{u^2}{2}}\,du = e^{\frac{\sigma^2 t^2}{2}}.$$

Therefore,

$$\mathbb{E}\, e^{tX} = e^{t\mu} \cdot e^{\frac{\sigma^2 t^2}{2}} = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$