

1 Improper Integrals

In this section, we will introduce the notion of integrals over intervals of infinite length or integrals of functions with an infinite discontinuity. These definite integrals are called *improper integrals*, and are understood as the limits of the integrals we introduced in Week 1.

Definition 1. We define two types of improper integrals:

1. **Infinite Region:** If f is continuous on $[a, \infty)$ or $(-\infty, b]$, the integral over an infinite domain is defined as the respective limit of integrals over finite intervals,

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx, \quad \int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx.$$

If both $\int_a^\infty f(x) dx < \infty$ and $\int_{-\infty}^a f(x) dx < \infty$, then

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx.$$

If one of the limits do not exist or is infinite, then $\int_{-\infty}^\infty f(x) dx$ *diverges*.

2. **Infinite Discontinuity:** If f is continuous on $[a, b)$ or $(a, b]$, the improper integral for a discontinuous function is defined as the respective limit of integrals over finite intervals,

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx \quad \int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx.$$

If f has a discontinuity at $c \in (a, b)$ and both $\int_a^c f(x) dx < \infty$ and $\int_c^b f(x) dx < \infty$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

If one of the limits do not exist or is infinite, then $\int_a^b f(x) dx$ *diverges*.

If the limit in the above definitions exist and is finite, then we say the integrals are *convergent*. If the limits do not exist, or if it equals $\pm\infty$, then we say the integral is *divergent*.

Example 1. An illustration of the two main types of improper integrals are depicted below.

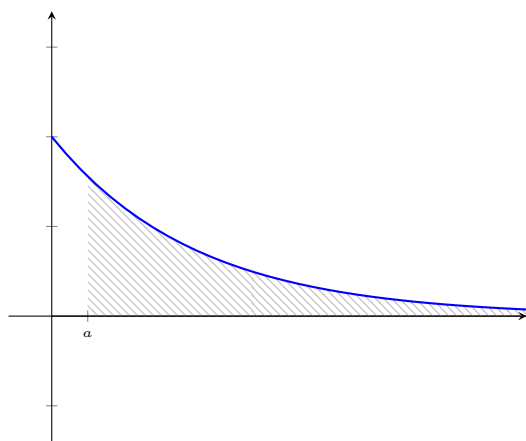


Figure 1: Infinite region case

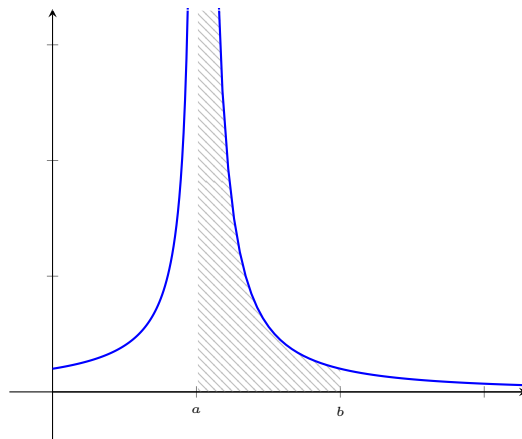


Figure 2: Infinite discontinuity case

1.1 Example Problems

Strategy: We compute the integrals normally then take a limit at the last step to evaluate it.

Problem 1. (★★) Find

$$\int_0^1 \ln(x) dx.$$

Solution 1. Since $\lim_{x \rightarrow 0} \ln(x) = -\infty$, the integral is interpreted as an improper integral. Computing the integral like usual, we see that

$$\begin{aligned} \int_0^1 \ln(x) dx &= \lim_{t \rightarrow 0^+} \int_t^1 \ln(x) dx \\ &= \lim_{t \rightarrow 0^+} \left(x \ln(x) - x \right) \Big|_{x=t}^{x=1} && \text{By Parts} \\ &= \lim_{t \rightarrow 0^+} \left(-1 - t \ln(t) + t \right) && \text{L'Hôpital's Rule} \\ &= -1. \end{aligned}$$

To compute the limit of $t \ln(t)$, we used L'Hôpital's Rule,

$$\lim_{t \rightarrow 0^+} t \ln(t) = \lim_{t \rightarrow 0^+} \frac{\ln(t)}{\frac{1}{t}} = \lim_{t \rightarrow 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t \rightarrow 0^+} -t = 0.$$

Problem 2. (★) Find

$$\int_1^{\infty} \frac{1}{x} dx.$$

Solution 2. Since we are integrating over an infinite region, the integral is interpreted as an improper integral. Computing the integral like usual, we see that

$$\begin{aligned} \int_1^{\infty} \frac{1}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t \rightarrow \infty} \ln(x) \Big|_{x=1}^{x=t} \\ &= \lim_{t \rightarrow \infty} \ln(t) && \ln(1) = 0 \\ &= \infty. \end{aligned}$$

Since the value is $+\infty$, this integral is divergent.

Problem 3. (★★) Find

$$\int_{-1}^2 \frac{1}{x^4} dx.$$

Solution 3. Since $\lim_{x \rightarrow 0^+} \frac{1}{x^4} = \infty$, the integral is interpreted as an improper integral,

$$\int_{-1}^2 \frac{1}{x^4} dx = \int_{-1}^0 \frac{1}{x^4} dx + \int_0^2 \frac{1}{x^4} dx$$

Computing the first integral like usual, we see that

$$\begin{aligned}\int_{-1}^0 \frac{1}{x^4} dx &= \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^4} dx \\ &= \lim_{t \rightarrow 0^-} \left. -\frac{1}{3}x^{-3} \right|_{x=-1}^{x=t} \\ &= \lim_{t \rightarrow 0^-} -\frac{1}{3} \cdot \frac{1}{t^3} - \frac{1}{3} \\ &= \infty.\end{aligned}$$

Since this integral diverges, $\int_{-1}^2 \frac{1}{x^4} dx$ also diverges.

Remark: It is very easy to make a mistake in this problem, because blindly integrating will give us

$$\int_{-1}^2 \frac{1}{x^4} dx = \left. -\frac{x^{-3}}{3} \right|_{x=-1}^{x=2} = -\frac{1}{24} - \frac{1}{3} = -\frac{3}{8}.$$

This is not correct, because the integral does not exist, because $\int_{-1}^0 \frac{1}{x^4} dx$ diverges. The error occurred because we incorrectly applied the fundamental theorem of calculus to a discontinuous function.

Problem 4. (★) Compute

$$\int_3^{\infty} \frac{1}{x(x-1)(x-2)} dx.$$

Solution 4. Since we are integrating over an infinite region, the integral is interpreted as an improper integral. Computing the integral like usual, we see that

$$\begin{aligned}\int_3^{\infty} \frac{1}{x(x-1)(x-2)} dx &= \lim_{t \rightarrow \infty} \int_3^t \frac{1}{x(x-1)(x-2)} dx \\ &= \lim_{t \rightarrow \infty} \int_3^t \left(\frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2} \right) dx \quad \text{Partial Fractions} \\ &= \lim_{t \rightarrow \infty} \left(\frac{1}{2} \ln x - \ln(x-1) + \frac{1}{2} \ln(x-2) \right) \Big|_{x=3}^{x=t} \\ &= \lim_{t \rightarrow \infty} \left(\frac{1}{2} \ln t - \ln(t-1) + \frac{1}{2} \ln(t-2) \right) - \left(\frac{1}{2} \ln 3 - \ln 2 + \frac{1}{2} \ln 1 \right).\end{aligned}$$

To compute the limit, notice that the first term is equal to

$$\lim_{t \rightarrow \infty} \ln \left(\frac{t^{1/2}(t-2)^{1/2}}{t-1} \right) = \ln \left(\lim_{t \rightarrow \infty} \left(\frac{t(t-2)}{(t-1)^2} \right)^{\frac{1}{2}} \right) = \frac{1}{2} \ln \left(\lim_{t \rightarrow \infty} \frac{t(t-2)}{(t-1)^2} \right) = 0.$$

We can conclude that

$$\int_3^{\infty} \frac{1}{x(x-1)(x-2)} dx = -\left(\frac{1}{2} \ln 3 - \ln 2 + \frac{1}{2} \ln 1 \right) = \frac{1}{2} \ln \left(\frac{4}{3} \right) \approx 0.143841.$$

2 Convergence Tests

We can determine the convergence of integrals without explicitly computing them by comparing them with simpler integrals.

2.1 Comparison Test

If $f(x) \geq g(x) \geq 0$, then the area under g is smaller than the area under f . Intuitively, if the bigger area is finite, then so is the smaller area, and if smaller area is infinite, then so is the bigger one. This intuition is made precise with a result called the comparison test:

Theorem 1 (Comparison Test). *If $0 \leq g(x) \leq f(x)$ for $x \geq a$ and f, g are continuous, then*

$$0 \leq \int_a^\infty g(x) dx \leq \int_a^\infty f(x) dx.$$

This means

1. *If $\int_a^\infty f(x) dx$ converges, then $\int_a^\infty g(x) dx$ also converges.*
2. *If $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ also diverges.*

Remark: A similar result holds for infinite functions. For example, if $\int_a^b f(x) dx$ converges and $0 \leq g(x) \leq f(x)$ for $x \in [a, b)$, then $\int_a^b g(x) dx$ is also convergent.

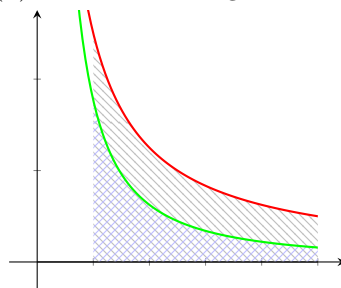


Figure 3: The area under the green curve is smaller than the area under the red curve. If the area under the red curve is finite, then the area under the green curve will also be finite.

2.2 Limit Comparison Test

Intuitively, the convergence of a function on an infinite region is determined by its values for large x . That is, if two continuous functions behave the same for large values of x , then both functions should converge or both functions should diverge.

Theorem 2 (Limit Comparison Test). *Suppose that $f(x)$ and $g(x)$ are positive and continuous for $x \geq a$, and*

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L \in (0, \infty).$$

Then $\int_a^\infty f(x) dx$ is convergent if and only if $\int_a^\infty g(x) dx$ is convergent.

2.3 Tails of Convergent Integrals

Intuitively, if an integral of a non-negative function over an infinite region is finite, then it must get small near ∞ . Likewise, if the tail of the function does not vanish at ∞ , then the integral must diverge.

Theorem 3. *If $f(x) \geq 0$ is continuous and $\int_a^\infty f(x) dx < \infty$, then*

$$\lim_{t \rightarrow \infty} \int_t^\infty f(x) dx = 0.$$

2.4 Example Problems

Strategy: To determine the convergence of an improper integral, we usually compare it to one of the following integrals:

1. The integral

$$\int_1^{\infty} \frac{1}{x^p} dx \quad (1)$$

converges for $p > 1$ and diverges for $p \leq 1$.

2. The integral

$$\int_0^1 \frac{1}{x^p} dx \quad (2)$$

converges for $p < 1$ and diverges for $p \geq 1$.

3. The integral

$$\int_0^{\infty} e^{-ax} dx \quad (3)$$

converges for $a > 0$.

Problem 1. (★) Is $\int_0^{\infty} e^{-\frac{x^2}{2}} dx$ convergent?

Solution 1. We first split the integral

$$\int_0^{\infty} e^{-\frac{x^2}{2}} dx = \int_0^1 e^{-\frac{x^2}{2}} dx + \int_1^{\infty} e^{-\frac{x^2}{2}} dx.$$

The first integral is a regular definite integral, so it is finite. We can use the comparison test to determine the convergence of the second integral. Since $\frac{x}{2} \leq \frac{x^2}{2}$ for $x \geq 1$, we have

$$e^{-\frac{1}{2}x} \geq e^{-\frac{x^2}{2}} \text{ for } x \geq 1 \implies \int_1^{\infty} e^{-\frac{x^2}{2}} dx \leq \int_1^{\infty} e^{-\frac{1}{2}x} dx.$$

The upper bounding integral $\int_1^{\infty} e^{-\frac{1}{2}x} dx$ converges (see equation (3)), so we can conclude $\int_1^{\infty} e^{-\frac{x^2}{2}} dx$ is convergent by the comparison test (the integral $\int_1^{\infty} e^{-\frac{x^2}{2}} dx$ is upper bounded by a finite number).

Remark: This makes sense intuitively because $e^{-\frac{1}{2}x^2}$ goes to zero way faster than any polynomial.

Problem 2. (★) Is $\int_2^{\infty} \frac{1}{\ln(x)} dx$ convergent?

Solution 2. Since $x \geq x - 1 \geq \ln(x)$ (draw the tangent line of $\ln(x)$ at $x = 1$), we have

$$\frac{1}{x} \leq \frac{1}{\ln(x)} \text{ for } x \geq 2 \implies \int_2^{\infty} \frac{1}{x} dx \leq \int_2^{\infty} \frac{1}{\ln(x)} dx.$$

The lower bounding integral $\int_2^{\infty} \frac{1}{x} dx$ diverges (see equation (1)), so we can conclude $\int_2^{\infty} \frac{1}{\ln(x)} dx$ is divergent by the comparison test (the integral $\int_2^{\infty} \frac{1}{\ln(x)} dx$ is lower bounded by infinity).

Remark: To guess the convergence, near ∞ we have $\ln(x) \ll x^\epsilon$ for all $\epsilon > 0$. This means that we can treat $\ln(x)$ as something that grows slower than any monomial. In particular, we have $\frac{1}{\ln(x)}$ goes to zero slower than a monomial, so it should not converge.

Problem 3. (★★) Is $\int_1^2 \frac{1}{\ln(x)} dx$ convergent?

Solution 3. Since $x - 1 \geq \ln(x)$ (draw the tangent line of $\ln(x)$ at $x = 1$), we have

$$\frac{1}{x-1} \leq \frac{1}{\ln(x)} \text{ for } x \geq 1 \implies \int_0^1 \frac{1}{u} du = \int_1^2 \frac{1}{x-1} dx \leq \int_1^2 \frac{1}{\ln(x)} dx.$$

The lower bounding integral $\int_0^1 \frac{1}{u} du$ diverges (see equation (2)), so we can conclude $\int_1^2 \frac{1}{\ln(x)} dx$ is divergent by the comparison test (the integral $\int_1^2 \frac{1}{\ln(x)} dx$ is lower bounded by infinity).

Problem 4. (★★) Is $\int_1^\infty \frac{3x+\sqrt{x}}{\sqrt{2x^7+2x+5}} dx$ convergent?

Solution 4. We will use the limit comparison test in this problem. Asymptotically, we have

$$\frac{3x + \sqrt{x}}{\sqrt{2x^7 + 2x + 5}} \sim \frac{3x}{\sqrt{2x^7}} = \frac{3}{\sqrt{2}} x^{-\frac{5}{2}}.$$

To see this, we can explicitly compute the limit of the ratio,

$$\lim_{x \rightarrow \infty} \frac{1}{\frac{3}{\sqrt{2}} x^{-\frac{5}{2}}} \cdot \frac{3x + \sqrt{x}}{\sqrt{2x^7 + 2x + 5}} = \lim_{x \rightarrow \infty} \frac{\sqrt{2}}{3} \cdot \frac{3x^{\frac{7}{2}} + x^3}{\sqrt{2x^7 + 2x + 5}} = 1.$$

Since $\int_2^\infty \frac{3}{\sqrt{2}} x^{-\frac{5}{2}} dx$ converges (see equation (1)), we can conclude $\int_1^\infty \frac{3x+\sqrt{x}}{\sqrt{2x^7+2x+5}} dx$ also converges.

Problem 5. (★★★) Find the values of a such that

$$\int_1^\infty (1+x^2)^{-\frac{n+1}{2}} x^a dx$$

converges.

Solution 5. We will use the limit comparison test in this problem. Asymptotically, we have

$$(1+x^2)^{-\frac{n+1}{2}} x^a \sim \frac{x^a}{x^{n+1}} = x^{a-n-1}.$$

To see this, we can explicitly compute the limit of the ratio,

$$\lim_{x \rightarrow \infty} \frac{(1+x^2)^{-\frac{n+1}{2}} x^a}{x^{a-n-1}} = \lim_{x \rightarrow \infty} \frac{(1+x^2)^{-\frac{n+1}{2}}}{x^{-n-1}} = \lim_{x \rightarrow \infty} \left(\frac{1}{x^2} + 1\right)^{-\frac{n+1}{2}} = 1.$$

Since $x^{a-n-1} = \frac{1}{x^{-a+n+1}}$ converges when $-a+n+1 > 1 \implies a < n$ (see equation (1)), our integral converges for $a < n$ by the limit comparison test.