1 Trigonometric Substitutions

In this section, we will use the change of variables formula in the "opposite" direction to compute the integrals of functions containing square roots. For nice enough functions $g(\theta)$ (such as differentiable bijective functions), we have the following change of variables formulas:

Indefinite Integral Version: Setting $x = g(\theta)$, we have

$$\int f(x)\,dx = \int f(g(\theta))g'(\theta)\,d\theta.$$

Definite Integral Version: Setting $x = g(\theta)$, we have

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(\theta))g'(\theta)\,d\theta.$$

1.1 Trigonometric Substitutions

Our choice of the change of variables depends on what expressions appear in our integrals:

Expression	Substitution	Domain	Trig Identity
$c^2 - d^2x^2$	$\frac{c}{d}\sin(\theta)$	$\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$c^2 + d^2x^2$	$\frac{c}{d}\tan(\theta)$	$\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$d^2x^2 - c^2$	$\frac{c}{d}\sec(\theta)$	$\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Remark: In general, the domains of these substitution should be chosen to ensure that these functions have well defined inverses. These particular domains were also chosen to ensure that the expressions are non-negative for all θ in the respective domains. These domains were chosen for convenience, because we don't want to worry about absolute values signs in our computations.

1.2 Example Problems

Strategy: If one of the three expressions $c^2 - d^2x^2$, $c^2 + d^2x^2$ or $c^2x^2 - d^2$ appears in the integral, then that is a good indication that we should use a trigonometric substitution.

1. Depending on the expression that appears, we choose the corresponding substitution in the table and proceed like usual (see Week 3).
2. If we have an indefinite integral, in the last step of the problem we draw a triangle to express our answer in terms of the original variables.

Remark: Because of our choice of domains, all the expressions are positive under the change of variables, so we will not write absolute value signs in the computations below. If we do not explicitly state the domain of the change of variables, we will always take it to be the standard domain given in the table above.

Problem 1. ($\star$) Find

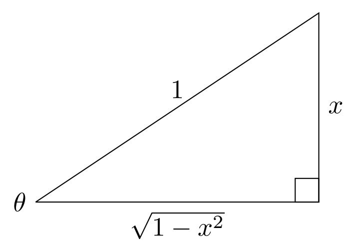
$$\int \sqrt{1 - x^2}\,dx$$

Solution 1.

Step 1: Since the term $1 - x^2$ appears in the problem, we should use the trigonometric substitution $x = \sin(\theta)$ where $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta && x = \sin\theta,\ dx = \cos\theta\,d\theta \\
&= \int \cos^2(\theta)\,d\theta && 1-\sin^2\theta = \cos^2\theta \\
&= \int \frac{1+\cos(2\theta)}{2}\,d\theta && \cos^2\theta = \frac{1+\cos(2\theta)}{2} \\
&= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C \\
&= \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2} + C. && \sin(2\theta) = 2\sin(\theta)\cos(\theta)
\end{aligned}$$

Step 2: We now need to write our answer in terms of x. The triangle corresponding to $x = \sin(\theta)$ is



From this triangle, we see that

$$\cos(\theta) = \sqrt{1-x^2}, \qquad \sin(\theta) = x \to \theta = \sin^{-1}(x),$$

which implies that

$$\int \sqrt{1-x^2}\,dx = \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2} + C = \frac{\sin^{-1}(x)}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

Remark: We implicitly assumed that $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ in the computations above. Technically, we should have written $\sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = |\cos(\theta)|$ as the intermediate step in our computation. However, since $\cos(\theta) \geq 0$ for $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, we could ignore this minor detail because of our clever choice of domain for trigonometric substitutions.

Problem 2. ($\star\star$) For $a, b > 0$, find

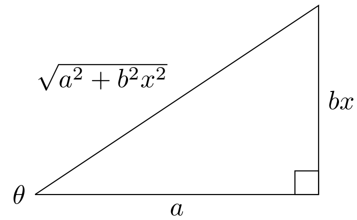
$$\int \frac{1}{\sqrt{b^2x^2+a^2}}\,dx$$

Solution 2.

Step 1: Since the term $a^2 + b^2x^2$ appears in the problem, we should use the trigonometric substitution $x = \frac{a}{b}\tan(\theta)$ where $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$,

$$\begin{aligned}
\int \frac{1}{\sqrt{b^2x^2+a^2}}\,dx &= \int \frac{a}{b}\frac{\sec^2\theta}{\sqrt{a^2\tan^2\theta + a^2}}\,d\theta && x = \frac{a}{b}\tan\theta,\ dx = \frac{a}{b}\sec^2\theta\,d\theta \\
&= \int \frac{1}{b}\sec(\theta)\,d\theta && 1+\tan^2\theta = \sec^2\theta \\
&= \frac{1}{b}\ln|\sec(\theta)+\tan(\theta)| + C. && \text{See Week 3 Question 6 on Page 4}
\end{aligned}$$

Step 2: We now need to write our answer in terms of x. The triangle corresponding to $\frac{bx}{a} = \tan(\theta)$ is



From this triangle, we see that

$$\sec(\theta) = \frac{\sqrt{a^2+b^2x^2}}{a}, \qquad \tan(\theta) = \frac{bx}{a},$$

which implies that

$$\int \frac{1}{\sqrt{b^2x^2+a^2}}\,dx = \frac{1}{b}\ln|\sec(\theta)+\tan(\theta)| + C = \frac{1}{b}\ln\left|\frac{\sqrt{a^2+b^2x^2}}{a} + \frac{bx}{a}\right| + C.$$

Problem 3. ($\star\star$) Find

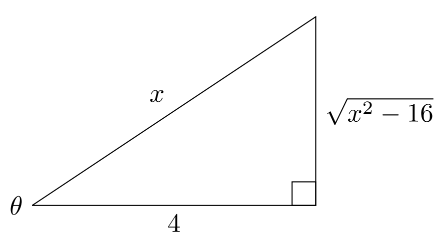
$$\int \frac{\sqrt{x^2-16}}{x^3}\,dx.$$

Solution 3.

Step 1: Since the term x^2-16 appears in the problem, we should use the trigonometric substitution $x = 4\sec(\theta)$ where $\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$,

$$\begin{aligned}
\int \frac{\sqrt{x^2-16}}{x^3}\,dx &= \int \frac{4\sqrt{16\sec^2\theta - 16}(\sec\theta+\tan\theta)}{4^3\sec^3\theta}\,d\theta && x = 4\sec\theta,\ dx = 4(\sec\theta\tan\theta)\,d\theta \\
&= \int \frac{\tan\theta(\sec\theta\tan\theta)}{4\sec^3\theta}\,d\theta && 1-\sec^2\theta = \tan^2\theta \\
&= \frac{1}{4}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\
&= \frac{1}{4}\int \sin^2\theta\,d\theta && \tan\theta = \frac{\sin\theta}{\cos\theta},\ \sec(\theta) = \frac{1}{\cos\theta} \\
&= \frac{1}{8}\int 1-\cos(2\theta)\,d\theta && \sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \\
&= \frac{1}{8}\theta - \frac{1}{16}\sin(2\theta) + C \\
&= \frac{1}{8}\theta - \frac{1}{8}\sin\theta\cos\theta + C. && \sin(2\theta) = 2\sin(\theta)\cos(\theta)
\end{aligned}$$

Step 2: We now need to write our answer in terms of x. The triangle corresponding to $\frac{x}{4} = \sec(\theta)$ is



From this triangle, we see that

$$\sin(\theta) = \frac{\sqrt{x^2-16}}{x}, \qquad \cos(\theta) = \frac{4}{x}, \qquad \sec(\theta) = \frac{x}{4} \to \theta = \sec^{-1}\left(\frac{x}{4}\right)$$

which implies that

$$\int \frac{\sqrt{x^2-16}}{x^3}\,dx = \frac{1}{8}\theta - \frac{1}{8}\sin\theta\cos\theta + C = \frac{1}{8}\sec^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}\frac{\sqrt{x^2-16}}{x^2} + C.$$

Remark: There are many possible definitions of $\sec^{-1}(x)$. The one we used in this problem corresponds to the inverse of $\sec(\theta)$ on our domain, i.e. $\sec^{-1}(x)$ has range $[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$.

Problem 4. ($\star$) Compute

$$\int_0^{\frac{3}{2}} \sqrt{9-4x^2}\,dx.$$

Solution 4. Since the term $9-4x^2$ appears in the problem, we should use the trigonometric substitution $x = \frac{3}{2}\sin(\theta)$ where $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned}
\int_0^{\frac{3}{2}} \sqrt{1-x^2}\,dx &= \int_0^{\frac{\pi}{2}} \frac{3}{2}\sqrt{9-9\sin^2\theta}\cos\theta\,d\theta && x = \frac{3}{2}\sin\theta,\ dx = \frac{3}{2}\cos\theta\,d\theta,\ \int_0^{\frac{3}{2}} dx \to \int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{9}{2}\int_0^{\frac{\pi}{2}} \cos^2(\theta)\,d\theta && 1-\sin^2\theta = \cos^2\theta \\
&= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\,d\theta && \cos^2\theta = \frac{1+\cos(2\theta)}{2} \\
&= \frac{9\theta}{4} + \frac{9\sin(2\theta)}{8}\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \\
&= \frac{9\pi}{8}.
\end{aligned}$$

Problem 5. ($\star\star\star$) Find

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx.$$

Solution 5.

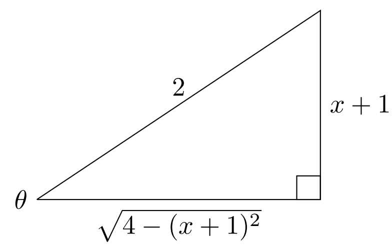
Step 1: It is not clear what trigonometric substitution to use immediately, so we first use algebra to simplify the integral first. By completing the square, we have

$$\frac{x}{\sqrt{3-2x-x^2}} = \frac{x}{\sqrt{3+1-1-2x-x^2}} = \frac{x}{\sqrt{4-(x+1)^2}}$$

Step 2: Since the term $4-(x+1)^2$ appears in the problem, we should use the trigonometric substitution $x+1 = 2\sin(\theta)$ where $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned}
\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx && \text{complete the square} \\
&= \int \frac{(2\sin\theta-1)2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta && x+1 = 2\sin\theta,\ dx = 2\cos\theta\,d\theta \\
&= \int 2\sin\theta - 1\,d\theta \\
&= -2\cos\theta - \theta + C.
\end{aligned}$$

Step 3: We now need to write our answer in terms of x. The triangle corresponding to $\frac{x+1}{2} = \sin(\theta)$ is



From this triangle, we see that

$$\cos(\theta) = \frac{\sqrt{4-(x+1)^2}}{2}, \qquad \sin(\theta) = \frac{x+1}{2},$$

which implies that

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = -2\cos\theta - \theta + C = -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$