1 Integration By Substitution (Change of Variables)

We can think of integration by substitution as the counterpart of the chain rule for differentiation. Suppose that $g(x)$ is a differentiable function and f is continuous on the range of g. Integration by substitution is given by the following formulas:

Indefinite Integral Version:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \qquad \text{where } u = g(x).$$

Definite Integral Version:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad \text{where } u = g(x).$$

1.1 Example Problems

Strategy: The idea is to make the integral easier to compute by doing a change of variables.

1. Start by guessing what the appropriate change of variable $u = g(x)$ should be. Usually you choose u to be the function that is "inside" the function.
2. Differentiate both sides of $u = g(x)$ to conclude $du = g'(x)dx$. If we have a definite integral, use the fact that $x = a \to u = g(a)$ and $x = b \to u = g(b)$ to also change the bounds of integration.
3. Rewrite the integral by replacing all instances of x with the new variable and compute the integral or definite integral.
4. If you computed the indefinite integral, then make sure to write your final answer back in terms of the original variables.

Problem 1. ($\star$) Find

$$\int \tan(x)\,dx.$$

Solution 1.

Step 1: We will use the change of variables $u = \cos(x)$,

$$\frac{du}{dx} = -\sin(x) \Rightarrow du = -\sin(x)\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos(x)| + C.$$

Problem 2. ($\star$) Find

$$\int_0^1 xe^{-\frac{x^2}{2}}\,dx.$$

Solution 2.

Step 1: We will use the change of variables $u = -\frac{x^2}{2}$,

$$\frac{du}{dx} = -x \Rightarrow du = -x\,dx, \quad x = 0 \to u = 0, \quad x = 1 \to u = -\frac{1}{2}.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int xe^{\frac{-x^2}{2}}\,dx = -\int_0^{-\frac{1}{2}} e^u\,du = -e^u\Big|_{u=0}^{u=-\frac{1}{2}} = -e^{-\frac{1}{2}} + 1.$$

Remark: Instead of changing the bounds of integration, we can first find the indefinite integral,

$$\int xe^{-\frac{x^2}{2}}\,dx = -e^{-\frac{x^2}{2}},$$

then use the fundamental theorem of calculus to conclude

$$\int_0^1 xe^{-\frac{x^2}{2}}\,dx = -e^{-\frac{x^2}{2}}\Big|_{x=0}^{x=1} = -e^{-\frac{1}{2}} + 1.$$

Problem 3. ($\star$) Find

$$\int \tanh(x)\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$$

Solution 3.

Step 1: We will use the change of variables $u = e^x + e^{-x}$,

$$\frac{du}{dx} = e^x - e^{-x} \Rightarrow du = (e^x - e^{-x})\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{du}{u} &= \ln|u| + C \\ &= \ln|e^x + e^{-x}| + C. \qquad u = e^x + e^{-x}\end{aligned}$$

Since $e^x + e^{-x} > 0$, we can remove the absolute values if we wish giving the final answer

$$\int \tanh(x)\,dx = \ln(e^x + e^{-x}) + C.$$

Remark: We can use the fact $e^x + e^{-x} = 2\cosh(x)$ to conclude that

$$\ln(e^x + e^{-x}) + C = \ln(2\cosh(x)) + C = \ln(\cosh(x)) + \underbrace{\ln(2) + C}_{D} = \ln(\cosh(x)) + D.$$

This form of the indefinite integral may be easier to remember since it mirrors the fact that

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C.$$

Problem 4. ($\star$) Evaluate

$$\int_0^1 x\sqrt{1-x^2}\,dx.$$

Solution 4.

Step 1: We will use the change of variables $u = 1 - x^2$,

$$\frac{du}{dx} = -2x \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = xdx, \qquad x = 0 \to u = 1, \qquad x = 1 \to u = 0.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=0} = \frac{1}{3}.$$

Remark: Instead of changing the bounds of integration, we can first find the indefinite integral,

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}},$$

then use the fundamental theorem of calculus to conclude

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

Problem 5. ($\star\star$) Find

$$\int \frac{1}{1+\sqrt{x}}\,dx.$$

Solution 5.

Step 1: We will use the change of variables $u = \sqrt{x}$,

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x}du = dx \Rightarrow 2u\,du = dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+u}\,du.$$

This integral is a bit tricky to compute, so we have to use algebra to simplify it first. Using long division to first simplify the integrand, we get

$$\begin{aligned}\int \frac{2u}{1+u}\,du = 2\int \frac{u}{1+u}\,du &= 2\int 1 - \frac{1}{1+u}\,du \\ &= 2u - 2\ln|1+u| + C \\ &= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + C. \qquad u = \sqrt{x}.\end{aligned}$$

Alternative Solution: We can also do a change of variables by writing x as a function of u.

Step 1: We can also do the change of variables $x = u^2$,

$$\frac{dx}{du} = 2u \Rightarrow dx = 2u\,du.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+\sqrt{u^2}}\,du = \int \frac{2u}{1+u}\,du.$$

The computation is now identical to the case above.

Problem 6. ($\star\star$) Find

$$\int \sec(x)\,dx.$$

Solution 6. We first do a trick by multiplying the numerator and denominator by $\sec(x)+\tan(x)$,

$$\int \sec(x)\,dx = \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx.$$

Step 1: We will use the change of variables $u = \sec(x)+\tan(x)$,

$$\frac{du}{dx} = \sec(x)\tan(x)+\sec^2(x) \Rightarrow du = (\sec(x)\tan(x)+\sec^2(x))\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \sec(x)\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x))}{\sec(x)+\tan(x)}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u|+C \\ &= \ln|\sec(x)+\tan(x)|+C. \qquad u=\sec(x)+\tan(x)\end{aligned}$$

Problem 7. ($\star\star$) Find

$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx.$$

Solution 7.

Step 1: We will use the change of variables $u = e^x$,

$$\frac{du}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}\,du \Rightarrow dx = \frac{1}{u}\,du.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx &= \int \frac{2}{u(u+u^{-1})}\,du \\ &= \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u)+C \\ &= 2\tan^{-1}(e^x)+C. \qquad u=e^x\end{aligned}$$

Alternative Solution: We first do a trick by multiplying the numerator and denominator by e^x,

$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x+e^{-x}}\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx.$$

Step 1: We will use the change of variables $u = e^x$,

$$\frac{du}{dx} = e^x \Rightarrow du = e^x\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx = \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u) + C \\ &= 2\tan^{-1}(e^x) + C. \qquad u = e^x\end{aligned}$$

1.1.1 Proofs of the Symmetry Properties of Integration

Problem 1. $(\star\star\star)$ Suppose that $f(-x) = f(x)$. Prove that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Solution 1. By the properties of definite integrals, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$

Using the change of variables $u = -x$ on the first integral, for even function f,

$$\begin{aligned}\int_0^{-a} f(x)\,dx &= -\int_0^a f(-u)\,du \qquad u=-x,\ du=-dx,\ x=0\to u=0,\ x=-a\to u=a \\ &= -\int_0^a f(u)\,du \qquad f(-x)=f(x) \\ &= -\int_0^a f(x)\,dx.\end{aligned}$$

This computation implies

$$\int_{-a}^{a} f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Problem 2. $(\star\star\star)$ Suppose that $f(-x) = -f(x)$. Prove that

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Solution 2. By the properties of definite integrals, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$

Using the change of variables $u = -x$ on the first integral, for odd functions f,

$$\begin{aligned}\int_0^{-a} f(x)\,dx &= -\int_0^a f(-u)\,du \qquad u=-x,\ du=-dx,\ x=0\to u=0,\ x=-a\to u=a \\ &= \int_0^a f(u)\,du \qquad f(-x)=-f(x) \\ &= \int_0^a f(x)\,dx.\end{aligned}$$

This computation implies

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

Problem 3. ($\star\star$) Suppose $f(x)$ is even ($f(-x) = f(x)$). Show that the function

$$F(x) = \int_{0}^{x} f(t)\,dt$$

is an odd function.

Solution 3. It suffices to show $F(-x) = -F(x)$. Using the change of variables $u = -t$,

$$du = -dt, \qquad t = 0 \to u = 0, \qquad t = -x \to u = x$$

we have

$$\begin{aligned} F(-x) = \int_{0}^{-x} f(t)\,dt &= -\int_{0}^{x} f(-u)\,du \\ &= -\int_{0}^{x} f(u)\,du \qquad f(-u) = f(u) \\ &= -F(x). \end{aligned}$$

Problem 4. ($\star\star$) Suppose $f(x)$ is odd ($f(-x) = -f(x)$). Show that the function

$$F(x) = \int_{a}^{x} f(t)\,dt$$

is an even function.

Solution 4. It suffices to show $F(-x) = F(x)$. Using the change of variables $u = -t$,

$$du = -dt, \qquad t = a \to u = -a, \qquad t = -x \to u = x$$

we have

$$F(-x) = \int_{a}^{-x} f(t)\,dt = -\int_{-a}^{x} f(-u)\,du = \int_{-a}^{x} f(u)\,du. \qquad f(-u) = -f(u)$$

It may appear that the last term is not of the same form as the term $F(x)$ because the lower bounds of integration are different. However, we can split the region of integration and use a change of variables to conclude that

$$\begin{aligned} \int_{-a}^{x} f(u)\,du &= \int_{-a}^{0} f(u)\,du + \int_{0}^{x} f(u)\,du \\ &= -\int_{a}^{0} f(-\tilde{u})\,d\tilde{u} + \int_{0}^{x} f(u)\,du \qquad \tilde{u} = -u, d\tilde{u} = -du, \int_{-a}^{0} du \to \int_{a}^{0} d\tilde{u} \\ &= \int_{a}^{0} f(\tilde{u})\,d\tilde{u} + \int_{0}^{x} f(u)\,du \qquad f(-u) = -f(u) \\ &= \int_{a}^{x} f(t)\,dt = F(x). \end{aligned}$$

Remark: If we use the result from Problem 2 on Page 5, then we have the shorter proof,

$$F(-x) = \int_{a}^{-x} f(t)\,dt = -\int_{-a}^{x} f(-u)\,du = \int_{-a}^{x} f(u)\,du = \underbrace{\int_{-a}^{a} f(u)\,du}_{=0} + \int_{a}^{x} f(u)\,du = F(x).$$

2 Integration By Parts

We can think of integration by substitution as the counterpart of the product rule for differentiation. Suppose that $u(x)$ and $v(x)$ are continuously differentiable functions. Integration by parts is given by the following formulas:

Indefinite Integral Version:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

Definite Integral Version:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_{x=a}^{x=b} - \int_a^b u'(x)v(x)\,dx.$$

2.1 Tabular Method

We will introduce a method to bookkeep multiple integration by parts steps simultaneously. This is called the tabular method for integration by parts. You pick a term to differentiate and a term to integrate then repeat the operation until the product of the terms in the last entry of the table is easy to integrate.

The integral can be recovered by multiplying diagonally across the rows of the table adding up all terms with alternating signs. The last term in the table is integrated across.

For example, the formula to integrate $\int u(x)v'''(x)\,dx$ by parts can be encoded by the table

$\pm$	D		I
$+$	u	$+$	v'''
$-$	u'	$-$	v''
$+$	u''	$+$	v'
$-\int$	u'''	$-\int$	v

which gives us the formula

$$\int u(x)v'''(x)\,dx = u(x)v''(x) - u'(x)v'(x) + u''(x)v(x) - \int u'''(x)v(x)\,dx.$$

2.2 Example Problems

Problem 1. ($\star$) Find

$$\int xe^x\,dx.$$

Solution 1.

Step 1: Draw the table

$\pm$	D	I
$+$	x	e^x
$-$	1	e^x
$+\int$	0	e^x

Step 2: From the table, we have

$$\int xe^x\,dx = xe^x - e^x + C.$$

Problem 2. $(\star\star)$ Find

$$\int x^6 e^x\,dx.$$

Solution 2.

Step 1: Draw the table

$\pm$	D	I
$+$	x^6	e^x
$-$	$6x^5$	e^x
$+$	$30x^4$	e^x
$-$	$120x^3$	e^x
$+$	$360x^2$	e^x
$-$	$720x$	e^x
$+$	720	e^x
$-\int$	0	e^x

Step 2: From the table, we have

$$\int x^6 e^x\,dx = x^6e^x - 6x^5e^x + 30x^4e^x - 120x^2e^x - 720xe^x + 720e^x + C.$$

Problem 3. $(\star\star)$ Find

$$\int x^4 \sin x\,dx.$$

Solution 3.

Step 1: Draw the table

$\pm$	D	I
$+$	x^4	$\sin x$
$-$	$4x^3$	$-\cos x$
$+$	$12x^2$	$-\sin x$
$-$	$24x$	$\cos x$
$+$	24	$\sin x$
$-\int$	0	$-\cos x$

Step 2: From the table, we have

$$\int x^4 \sin x\,dx = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

Problem 4. ($\star\star$) Find

$$\int e^x \sin x\,dx.$$

Solution 4.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin x$	e^x
$-$	$\cos x$	e^x
$+\int$	$-\sin x$	e^x

Step 2: From the table, we have

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx + D.$$

Moving all the $\int e^x \sin x\,dx$ to one side and simplifying, we can conclude

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x + D \implies \int e^x \sin x\,dx = \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x + C.$$

Problem 5. ($\star\star\star$) Find

$$\int xe^x \cos(x)\,dx.$$

Solution 5.

Step 1: Draw the table

$\pm$	D	I
$+$	$x\cos x$	e^x
$-$	$\cos x - x\sin x$	e^x
$+\int$	$-2\sin x - x\cos x$	e^x

Step 2: From the table, we have

$$\int xe^x \cos x\, dx = xe^x \cos x - e^x \cos x + xe^x \sin x - 2\int e^x \sin x\, dx - \int xe^x \cos x\, dx.$$

Moving all the $\int xe^x \cos x\, dx$ to one side and simplifying, we can conclude

$$\begin{aligned} 2\int xe^x \cos x\, dx &= xe^x \cos x - e^x \cos x + xe^x \sin x - 2\int e^x \sin x\, dx \\ &= xe^x \cos x - e^x \cos x + xe^x \sin x - e^x \sin x + e^x \cos x + C. \end{aligned}$$

Problem 4

Dividing both sides by 2, we can conclude

$$\int xe^x \cos x\, dx = \frac{1}{2}\Big(xe^x \cos x + xe^x \sin x - e^x \sin x\Big) + C.$$

Problem 6. (⋆) Find

$$\int \ln(x)\, dx.$$

Solution 6.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln(x)$	1
$-\int$	$\frac{1}{x}$	x

Step 2: From the table, we have

$$\int \ln(x)\, dx = x\ln(x) - \int 1\, dx = x\ln(x) - x + C.$$

Problem 7. (⋆⋆) Evaluate

$$\int_1^2 x^3 \ln x\, dx.$$

Solution 7.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln x$	x^3
$-\int$	$\frac{1}{x}$	$\frac{1}{4}x^4$

Step 2: From the table, we have

$$\int x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$$

Step 3: We can now use the fundamental theorem of calculus to compute the definite integral,

$$\int_1^2 x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\Big|_{x=1}^{x=2} = 4\ln 2 - 1 + \frac{1}{16} = 4\ln 2 - \frac{15}{16}.$$

Problem 8. ($\star\star\star$) Derive the reduction formula

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.$$

Solution 8.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin^{n-1}(x)$	$\sin(x)$
$-\int$	$(n-1)\cos(x)\sin^{n-2}(x)$	$-\cos(x)$

Step 2: From the table, we have

$$\begin{aligned}\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x) \qquad \sin^2(x)+\cos^2(x)=1\\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx\end{aligned}$$

Moving all the the $\int \sin^n(x)\,dx$ terms to one side, we can conclude

$$\begin{aligned} n\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx\\ \Rightarrow \int \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.\end{aligned}$$

Problem 9. ($\star\star$) For $x \in \mathbb{R}$, the *error function* is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

Find

$$\int \operatorname{erf}(x)\,dx.$$

Solution 9. We can integrate by parts,

$\pm$	D	I
$+$	$\operatorname{erf}(x)$	1
$-\int$	$\frac{d}{dx}\operatorname{erf}(x)$	x

Since $\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$ by the fundamental theorem, we have

$$\int \operatorname{erf}(x)\,dx = x\operatorname{erf}(x) - \int \frac{2x}{\sqrt{\pi}}e^{-x^2}\,dx.$$

The second integral can be solved using the substitution $u = -x^2$, $du = -2xdx$ which gives us

$$\int \operatorname{erf}(x)\,dx = x\operatorname{erf}(x) + \int \frac{1}{\sqrt{\pi}}e^u\,du = x\operatorname{erf}(x) + \frac{1}{\sqrt{\pi}}\cdot e^{-x^2} + C.$$

Remark: It is easy to check that the $x\operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + C$ is an antiderivative by simply differentiating.