1 Taylor Series

We want to represent functions using power series. Polynomials are the easiest functions understand, so power series expansions can be used to understand the behavior of more complicated functions.

Definition 1. The *Taylor series* of $f(x)$ at a is a power series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots. \qquad (1)$$

The *kth-degree Taylor polynomial* is the partial sum up to the kth term,

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The polynomial $P_k(x)$ only has terms up to degree k. If the partial sums of the power series converge to $f(x)$, then we can interpret the Taylor polynomials as polynomial approximations of $f(x)$. Notice that $P_1(x)$ is the usual linear approximation of $f(x)$ at $x = a$. The Taylor polynomials are essentially high order approximations of $f(x)$. The derivatives of $P_k(x)$ agree with the derivatives of $f(x)$ at $x = a$ up to order k, so the behavior near $x = a$ is closely approximated by $P_k(x)$.

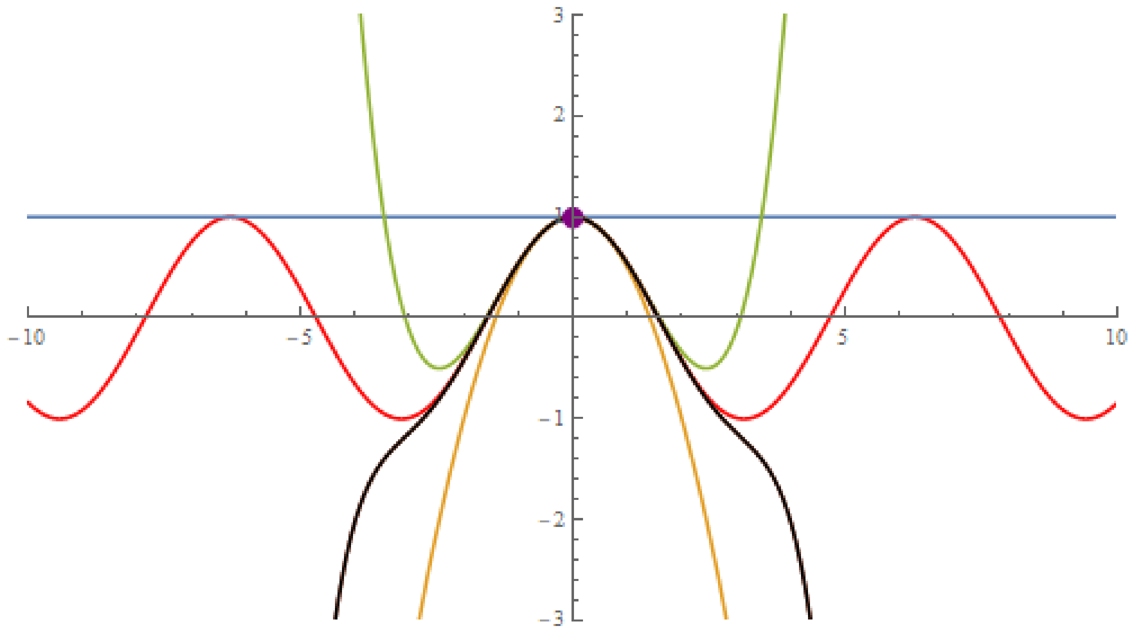

Figure 1: The Taylor polynomials $P_0(x)$, $P_2(x)$, $P_4(x)$ and $P_6(x)$ of $\cos(x)$ around $x = 0$ is displayed above. As the degree of the Taylor polynomials increase, the approximations become more accurate. The figure suggests that Taylor polynomials approximations are good local approximations of $\cos(x)$.

If f is *analytic*, then f agrees with its Taylor series on its interval of convergence. Even though the figure might suggest that Taylor approximations are only good for x near a, the approximation can be valid very far away from a provided we take a polynomial of high enough order. The difference $R_k(x)$ between the approximation $P_k(x)$ and its exact value $f(x)$ is called the *error* or *remainder*

$$R_k(x) = f(x) - P_k(x).$$

A function $f(x)$ equals its Taylor series at x only when $\lim_{k\to\infty} R_k(x) = 0$.

1.1 Important Taylor Series and its Radius of Convergence

Even though we can write down the formal Taylor series, it does not necessarily mean that the infinite series agrees with $f(x)$ everywhere. The radius of convergence tells us that $f(x)$ agrees with its Taylor series whenever $|x-a|<R$. A collection of important Taylor series and the corresponding radius of convergence is listed below:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad R = 1 \tag{2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty \tag{3}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty \tag{4}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty \tag{5}$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1 \tag{6}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad R = 1 \tag{7}$$

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \qquad R = 1 \tag{8}$$

Remark: Formulas (2), (3), (4), (5) are the important ones to memorize for this course.

1.2 Calculations with Power Series

Using the formula for the Taylor series can be very cumbersome. We can use existing power series to derive the formula for other power series. Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has radius of convergence R. On the interval $(a-R, a+R)$ we can manipulate infinite polynomials the same we manipulate finite polynomials:

1. *Term by term differentiation*: For $|x-a|<R$
$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}.$$
2. *Term by term integration*: For $|x-a|<R$,
$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}.$$
3. *Composition*: For $|h(x)-a|<R$,
$$f(h(x)) = \sum_{n=0}^{\infty} c_n (h(x)-a)^n.$$
4. *Addition, Multiplication, Division*: Identical to how we manipulate polynomials.

1.3 Approximation Errors

Infinite series are somewhat cumbersome to work with, so it will be nice if we can approximate f with the first few terms of the infinite sum. The resulting error approximating $f(x)$ with the kth order Taylor polynomial $P_k(x)$ is called the remainder term $R_k(x)$,

$$R_k(x) = f(x) - P_k(x) \iff f(x) = P_k(x) + R_k(x)$$

Taylor's Theorem gives an explicit formula for this remainder term. Taylor's Theorem can also be used to determine if the Taylor series converges to the function we are interested in.

Theorem 1 (Taylor's Theorem)**.** *Suppose that f^{k+1} is continuous in an open interval I containing a.*

1. *Lagrange Form: For every $x \in I$, there exists a some c between x and a such that*

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}. \tag{9}$$

2. *Integral Form: For every $x \in I$, we have*

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt. \tag{10}$$

Proof. We will use integration by parts and the fundamental theorem of calculus to prove (10). If f^{k+1} is continuous in an open interval I containing a, then the fundamental theorem of calculus and integration by parts applies to $f^{(n)}(x)$ on the region between x and a for all $0 \leq n \leq k+1$.

(a) The fundamental theorem of calculus implies

$$\int_a^x f'(t)\, dt = f(x) - f(a) \implies f(x) = f(a) + \int_a^x f'(t)\, dt,$$

verifying (10) when $k = 0$.

(b) Starting from the case $k = 0$ and integrating by parts (we differentiate $f'(t)$ and choose $t - x$ to be the antideriative of 1 in the integration by parts formula), we see that

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t)\, dt \\ &= f(a) + (t-x)f'(t)\Big|_{t=a}^{t=x} - \int_a^x f''(t)(t-x)\, dt \\ &= f(a) + (x-a)f'(t) + \int_a^x f''(t)(x-t)\, dt \end{aligned}$$

verifying (10) when $k = 1$.

(c) In general, we can use integration by parts to conclude that

$$\begin{aligned} \int_a^x \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}\, dt &= -\frac{f^{(k)}(t)}{k!}(x-t)^k\Big|_{t=a}^{t=x} + \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\, dt \\ &= \frac{f^{(k)}(t)}{k!}(x-a)^k + \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\, dt. \end{aligned}$$

The formula in (10) follows immediately by induction. □

Remark: If $k = 0$, then (9) is the mean value theorem and (10) is the fundamental theorem of calculus. Therefore, we can think of Taylor's Theorem as an extension of the mean value theorem or the fundamental theorem of calculus to higher derivatives. Equation (9) is proved in an exercise.

1.3.1 Special Case: Alternating Series

Sometimes the Taylor series form an alternating series with increments decreasing to 0. If this is the case, then we can estimate the error with the next term in the series,

Theorem 2 (Error Estimation for Alternating Series)**.** *If $(a_n)_{n\geq 1}$ is a sequence of positive numbers such that $(a_n)_{n\geq m}$ is decreasing and $\lim_{n\to\infty} a_n = 0$, then*

$$\left|\sum_{k=m}^{\infty}(-1)^k a_k\right| \leq a_m \qquad \textit{and} \qquad \left|\sum_{k=m}^{\infty}(-1)^{k+1} a_k\right| \leq a_m.$$

1.4 Example Problems

1.4.1 Taylor Series Formula

Problem 1. $(\star)$ Find the Taylor series for e^x at $x = 0$.

Solution 1. Let $f(x) = e^x$. Since $f^{(n)} = e^x$ for all n, the formula for the Taylor series (1) at $x = 0$ implies that

$$e^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty}\frac{e^0}{n!}x^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Problem 2. $(\star)$ Find the Taylor series for $\ln(1+x)$ at $x = 0$.

Solution 2. Let $f(x) = \ln(1+x)$. We have

$$f(0) = 0,\ f'(x) = \frac{1}{1+x},\ f''(x) = -\frac{1}{(1+x)^2},\ f^{(3)}(x) = \frac{2}{(1+x)^3},\ \ldots,\ f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

The formula for the Taylor series (1) at $x = 0$ implies that

$$\ln(1+x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n-1)!}{n!}x^n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

1.4.2 Power Series Calculations

Problem 1. $(\star)$ Find the Taylor series for

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

at $x = 0$. What is the radius of convergence?

Solution 1. Since $\cosh(x) = \frac{e^x+e^{-x}}{2} = \frac{1}{2}(e^x+e^{-x})$, we can use (3), the Taylor series of e^x, to conclude

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}(e^x + e^{-x}) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) && e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(1+(-1)^n)\cdot x^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} && (1+(-1)^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.}\end{cases}
\end{aligned}$$

Since e^x has radius of convergence $R = \infty$, so does $\cosh(x)$.

Problem 2. ($\star\star$) Find the Taylor series for $\arctan(x)$ at $x = 0$. What is the radius of convergence?

Solution 2. Since $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, it suffices to find the Taylor series of $\frac{1}{1+x^2}$ and integrate term by term. We can use (2), the Taylor series of $\frac{1}{1-x}$, with x replaced with $-x^2$ to conclude

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Integrating this term by term implies that

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

Since $\arctan(0) = 0$, we can evaluate both sides at $x = 0$ to conclude that $C = 0$, which implies that

$$\arctan(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

Since the Taylor series of $\frac{1}{1-(-x^2)}$ holds for $|-x^2| < 1$, the Taylor series for $\arctan(x)$ holds for $|x| < 1$.

Problem 3. ($\star$) Find the Taylor series for $\frac{1}{(1-x)^2}$ at $x = 0$. What is the radius of convergence?

Solution 3. Since $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$, it suffices to find the Taylor series of $\frac{1}{1-x}$ differentiate term by term. From (2), we know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n.$$

Differentiating this term by term implies that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \sum_{n=0}^{\infty}nx^{n-1}.$$

Since (2) holds for $|x| < 1$, we can conclude that our formula holds for $|x| < 1$.

Problem 4. ($\star$) Let $p \in (0, 1)$. Find

$$p\sum_{k=1}^{\infty}k(1-p)^{k-1}.$$

Solution 4. Since $|1-p| < 1$, we can use the formula in Problem 3 evaluated at $x = (1-p)$ to see

$$p\sum_{k=1}^{\infty}k(1-p)^{k-1} = p\sum_{k=0}^{\infty}k(1-p)^{k-1} = p\cdot\frac{1}{(1-(1-p))^2} = \frac{1}{p}.$$

Remark: The power series approach is much easier than the direct computation using double sums in Week 10 Problem 4 on page 5. The sum we computed in this problem is the expected value of a geometric random variable. If we take $p = 0.5$, then it says that on average, we need to flip a coin $\frac{1}{p} = \frac{1}{\frac{1}{2}} = 2$ times before we see the first heads.

Problem 5. ($\star\star\star$)

1. Find the first three non-zero terms of the Taylor series of $\tan(x)$ at $x = 0$.
2. Find the first three non-zero terms of the Taylor series of $\ln(\cos(x))$ at $x = 0$.

Solution 5.

(a) Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we can use (4) and (5), the Taylor series of $\sin(x)$ and $\cos(x)$, and polynomial long division to find the coefficients of $\tan(x)$. By polynomial long division,

$$\frac{x - \frac{x^3}{6} + \frac{x^5}{120} + \dots}{1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots.$$

The coefficients are equal to $-\frac{1}{6} + \frac{1}{2} = \frac{1}{3}$ and $\frac{1}{120} - \frac{1}{24} + \frac{1}{6} = \frac{2}{15}$. Notice that $\tan(x)$ is odd, so only odd powers appear in its Taylor series. Therefore,

$$\tan(x) \approx x + \frac{x^3}{3} + \frac{2x^5}{15}.$$

(b) Since $\ln(\cos(x)) = \int -\tan(x)\,dx$ we can integrate the first three terms of $\tan(x)$ in part (a) to conclude that

$$\int -x - \frac{x^3}{3} + \frac{2x^5}{15}\,dx = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + C.$$

Since $\ln(\cos(0)) = 0$, we can evaluate both sides at $x = 0$ to conclude that $C = 0$. Therefore,

$$\ln(\cos(x)) \approx -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45}.$$

1.4.3 Error Approximations

Problem 1. ($\star\star$) Find a polynomial approximation for $\sin(x)$ such that the maximum approximation error is less than 0.001 for values of x in $[-\pi/2, \pi/2]$.

Solution 1. We will use two methods to find a bound on the error.

Lagrange Bound: Consider the kth degree Taylor polynomial approximation of $\sin(x)$ at $x = 0$,

$$P_{2k+1}(x) = \sum_{n=0}^{k} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Recall that the Lagrange form of the remainder (at $a = 0$) is given by

$$R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1}.$$

We want to find a k such that

$$\left| \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1} \right| \le 0.001$$

We will find a rough bound for k. Since the derivatives of $\sin(x)$ are bounded by 1, if $|x| \le \frac{\pi}{2}$,

$$\left| \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1} \right| \le \left| \frac{1}{(k+1)!} \cdot \left(\frac{\pi}{2}\right)^{k+1} \right|.$$

If we take $k = 7$, then
$$|R_7(x)| = \left|\frac{1}{8!} \cdot \left(\frac{\pi}{2}\right)^8\right| \approx 0.00092 < 0.001.$$
This means that
$$|\sin(x) - P_7(x)| = |R_7(x)| < 0.001,$$
so
$$P_7(x) = \sum_{n=0}^{3} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$
accurately approximates $\sin(x)$ to within ± 0.001 for $x \in [-\pi/2, \pi/2]$.

Alternating Series Bound: If $x \in [-\pi/2, \pi/2]$, the Taylor series
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
is an alternating series with increments decreasing to 0, so the error is bounded by the next term in the approximation. If we consider a 7th order polynomial, then for $x \in [-\pi/2, \pi/2]$,
$$|R_7(x)| = \left|\sum_{n=4}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right| \le \left.\frac{|x|^{2n+1}}{(2n+1)!}\right|_{n=4} = \frac{|x|^9}{9!} \le \frac{\pi^9}{2^9 \cdot 9!} \approx 0.0002 < 0.001.$$
This means that $P_7(x) = \sum_{n=0}^{3} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ accurately approximates $\sin(x)$ to within ± 0.001 for $x \in [-\pi/2, \pi/2]$. Notice that this bound gave a slightly better bound on the error.

Problem 2. $(\star\star\star)$ Find the Taylor series approximation of
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
at $x = 0$. How accurate is the kth degree Taylor approximation?

Solution 2. It turns out that $f(x)$ is infinitely differentiable at $x = 0$. For $x \neq 0$, we can use the chain rule to compute its derivative explicitly,
$$f'(x) = \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}}, \quad f''(x) = \frac{4 - 6x^2}{x^6} \cdot e^{-\frac{1}{x^2}}, \quad f'''(x) = \frac{24x^4 - 36x^2 + 8}{x^9} \cdot e^{-\frac{1}{x^2}}, \quad \dots$$
In general, the nth derivative will be a rational functions up to order x^{-3n} times $e^{-\frac{1}{x^2}}$. If we take the limit as $x \to 0$ in each of the terms above, the $e^{-\frac{1}{x^2}}$ goes to 0 faster than x^{-3n} for all $n > 0$, so
$$f^{(n)}(0) = 0 \qquad \text{for all } n \ge 0.$$
This means that the Taylor series for $f(x)$ is the constant function 0. For any $k > 0$, the kth degree Taylor polynomial $P_k(x) = 0$, which implies that
$$R_k(x) = f(x) - P_k(x) = f(x) \neq 0$$
unless $x = 0$. Therefore, the Taylor polynomial approximation is completely useless for computing $f(x)$.

Remark: The Taylor series for f exists at $x = 0$, because $f(x)$ is infinitely differentiable there. Unfortunately, the Taylor series does not equal to $f(x)$ at any point except $x = 0$. This problem demonstrates that the Taylor series of a function might exist and converge everywhere, but it might not converge to the original function we wanted approximate.

1.4.4 Proofs of Error Approximation Theorems

Problem 1. ($\star\star\star$) Derive the Lagrange form of the remainder (9) assuming that the integral form (10) holds. In particular, if f^{k+1} is continuous in an open interval I containing a, find a $c \in I$ such that

$$\int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}.$$

Solution 1. If g is continuous and h does not change sign on $[a,b]$, then a generalization of the mean value theorem for integration states that there exists a $c \in [a,b]$ such that

$$\int_a^b g(x)h(x)\, dx = g(c)\int_a^b h(x)\, dx.$$

Since f^{k+1} is continuous in an open interval I containing a, and $(x-t)^k$ does not change sign for t between a and x, the generalized mean value theorem for integration states that there exists a c between a and x contained in I such that

$$\int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt = f^{(k+1)}(c)\int_a^x \frac{1}{k!}(x-t)^k \, dt = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}.$$

Remark: Since f^{k+1} is continuous in an open interval I containing a, the usual mean value theorem for integration implies there exists a $c \in I$ such that

$$\frac{1}{x-a}\int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt = \frac{f^{(k+1)}(c)}{k!}(x-c)^k.$$

Rearranging gives us the *Cauchy form* of the remainder,

$$\int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \, dt = \frac{f^{(k+1)}(c)}{k!}(x-c)^k(x-a).$$

Problem 2. ($\star\star\star$) Derive the error estimate for alternating series. That is, if $(a_n)_{n\geq 1}$ is a sequence of positive numbers such that $(a_n)_{n\geq m}$ is decreasing and $\lim_{n\to\infty} a_n = 0$, show that

$$\Big|\sum_{k=m}^{\infty}(-1)^k a_k\Big| \leq a_m \qquad \text{and} \qquad \Big|\sum_{k=m}^{\infty}(-1)^{k+1} a_k\Big| \leq a_m.$$

Solution 2. We can factor out a (-1) to conclude that $|\sum_{k=m}^{\infty}(-1)^k a_k| = |\sum_{k=m}^{\infty}(-1)^{k+1}a_k|$. Therefore, without loss of generality, we may assume that the coefficient of the a_m term in the sum is positive. Since $(a_n)_{n\geq m}$ is decreasing, for all $k \geq m$ we can conclude

$$a_{k+1} \geq a_{k+2} \implies a_k - a_{k+2} \geq a_k - a_{k+1} \geq 0 \implies a_k \geq a_k - a_{k+1} + a_{k+2} \geq 0.$$

We can apply this bound repeatedly, first to a_m then to a_{m+2} then to a_{m+4}, etc to conclude

$$a_m \geq a_m - a_{m+1} + a_{m+2} \geq a_m - a_{m+1} + a_{m+2} - a_{m+3} + a_{m+4} \geq \cdots \geq a_m - a_{m+1} + \cdots + a_{m+2k}.$$

This holds for all $k \geq 0$, so taking $k \to \infty$ implies that

$$a_m \geq \Big|\sum_{k=m}^{\infty}(-1)^k a_k\Big| = \Big|\sum_{k=m}^{\infty}(-1)^{k+1}a_k\Big|.$$