1 Infinite Sequences

An infinite sequence $(s_n)_{n\geq 1}$ is an infinite list of numbers,

$$s_1, s_2, s_3, \dots, s_n, \dots$$

Sometimes we will define a sequence by giving it a algebraic formula for the nth term. For example, the sequence $s_n = f(n)$ corresponds to the infinite list of numbers

$$f(1), f(2), f(3), \dots, f(n), \dots$$

Example 1. The sequence $(2n)_{n\geq 1}$ corresponds to the list of *even numbers*,

$$2, 4, 6, 8, \dots$$

Example 2. The sequence $s_n = \frac{1}{n}$ corresponds to the list of *harmonic numbers*,

$$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$$

Example 3. The recursive sequence $(s_n)_{n\geq 1}$ given by $s_0 = 0$, $s_1 = 1$, and $s_n = s_{n-1} + s_{n-2}$ for $n > 1$ corresponds to the *Fibonacci Sequence*,

$$1, 1, 2, 3, 5, 8, \dots$$

Example 4. If $(a_n)_{n\geq 1}$ is a sequence of numbers, then the sequence $s_n = \sum_{i=1}^{n} a_i$ corresponds to the sequence of *partial sums*,

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, \sum_{i=1}^{n} a_i, \dots$$

Definition 1. The notation

$$\lim_{n\to\infty} s_n = L$$

means that s_n gets arbitrarily close to L when n is sufficiently large.

1. If the limit L exists and is finite, then we say that the sequence $(s_n)_{n\geq 1}$ *converges*.
2. If L does not exist or is infinite, then we say that $(s_n)_{n\geq 1}$ *diverges*.
3. If $L = \pm\infty$, then we sometimes say the sequence $(s_n)_{n\geq 1}$ *diverges to infinity* to differentiate it from the case that L does not exist.

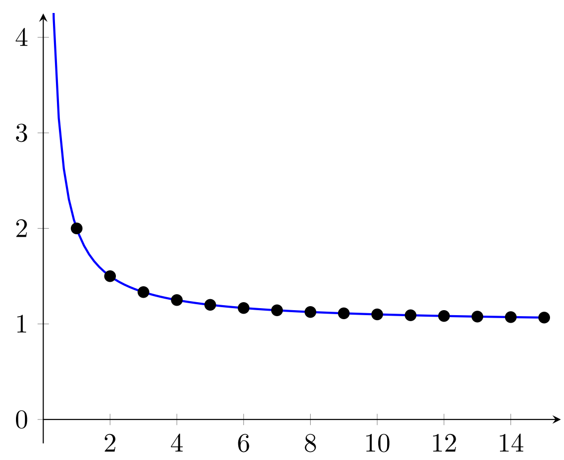

Figure 1: The sequence $s_n = f(n)$ can be thought of as the restriction of $f(x)$ to $x \in \mathbb{N}$. Therefore, the same rules we used to find limits of functions also apply to sequences.

The following convergence theorem for sequences will be used many times next week,

Theorem 1 (Monotone Convergence). *If $(s_n)_{n\geq 1}$ is monotone and bounded then it also converges.*

1.1 Example Problems

The limit laws for functions also hold for sequences s_n, so we can use the same tricks to compute the limits of sequences. If $\lim_{x\to\infty} f(x) = L$, then the sequence $s_n = f(n)$ also has limit L.

Problem 1. ($\star$) Determine whether the sequence $s_n = \frac{1}{n}$ converges or diverges. If the sequence converges, find its limit.

Solution 1. Since $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, $(s_n)_{n\geq 1}$ converges to 0.

Problem 2. ($\star$) Determine whether the sequence $s_n = \frac{n}{\ln n}$ converges or diverges. If the sequence converges, find its limit.

Solution 2. By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \infty.$$

Therefore, $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$, so $(s_n)_{n\geq 1}$ diverges to ∞.

Problem 3. ($\star\star\star$) Consider the recursively defined sequence $s_1 = 2$ and

$$s_n = \frac{1}{2}(s_{n-1} + 1) \qquad \text{for } n > 1.$$

Determine whether $(s_n)_{n\geq 1}$ converges or diverges. If the sequence converges, find its limit.

Solution 3. Since $s_2 = \frac{1}{2}(2+1) = \frac{3}{2}$, it is clear that $s_2 \leq s_1$. If we assume that $s_n \leq s_{n-1}$, then

$$s_n \leq s_{n-1} \implies \frac{1}{2}(s_n + 1) \leq \frac{1}{2}(s_{n-1} + 1) \implies s_{n+1} \leq s_n,$$

so $(s_n)_{n\geq 1}$ is decreasing by induction. Furthermore, $s_n \geq 0$ so it is bounded below and therefore convergent by the monotone convergence theorem. To compute the limit, we can assume that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = L$ and take the limit as $n \to \infty$ on both sides of the recurrence relation,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}(s_{n-1} + 1) \implies L = \frac{1}{2}(L+1) \implies L = 1.$$

Problem 4. ($\star\star$) Determine whether the sequence $s_n = \frac{n!}{n^n}$ converges or diverges. If the sequence converges, find its limit.

Solution 4. The sequence converges because

$$\frac{s_{n+1}}{s_n} = \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)!}{n!} = \left(\frac{n}{n+1}\right)^n < 1$$

which implies $s_{n+1} < s_n$, so the sequence is decreasing. Furthermore, $s_n \geq 0$ so it is bounded from below and therefore convergent by the monotone convergence theorem. This limit can be computed explicitly using the squeeze theorem. Since $\frac{k}{n} \leq 1$ for all $k \leq n$,

$$0 \leq \frac{n!}{n^n} = \frac{1\cdot 2 \cdots n}{n \cdot n \cdots n} = \frac{1}{n} \cdot \frac{2}{n} \cdots 1 \leq \frac{1}{n} \implies 0 \leq \lim_{n\to\infty} \frac{n!}{n^n} \leq \lim_{n\to\infty} \frac{1}{n} = 0 \implies \lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Problem 5. ($\star\star$) Find an example of a function such that the sequence $s_n = f(n)$ converges, but $\lim_{x\to\infty} f(x)$ does not exist.

Solution 5. A basic example is the function $f(x) = \sin(\pi x)$. It is easy to see that $\lim_{x\to\infty} f(x)$ does not exist because $f(x)$ oscillates between -1 and 1. However, $f(n) = \sin(n\pi) = 0$ for all n, so $(s_n)_{n\geq 1}$ is a sequence of 0's, which obviously converges.

2 Infinite Series

An infinite series is the sum of all the terms in a sequence $(a_n)_{n\geq 0}$,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \dots$$

The infinite series is interpreted as the limit of the sequence of its mth partial sums, $s_m = \sum_{n=0}^{m} a_n$,

$$\sum_{n=0}^{\infty} a_n = \lim_{m\to\infty} \sum_{n=0}^{m} a_n = \lim_{m\to\infty} s_m.$$

The same terminology for sequences also applies to series:

1. If $\lim_{m\to\infty} \sum_{n=0}^{m} a_n$ exists and is finite, then we say that the series $\sum_{n=0}^{\infty} a_n$ *converges.*
2. If $\lim_{m\to\infty} \sum_{n=0}^{m} a_n$ does not exist or is infinite, then we say say that the series $\sum_{n=0}^{\infty} a_n$ *diverges.*

Example 5. The *geometric series* is a series of the form

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + \dots$$

If $|x| < 1$, then the series converges and given explicitly by

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}.$$

The series diverges when $|x| \geq 1$.

Example 6. The *power series* (or power series centered at a) is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1 x + c_2 x^2 + \dots$$

The sequence $(c_n)_{n\geq 0}$ are called the coefficients of the power series. The *radius of convergence* is the largest number R such that

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

converges for all $|x - a| < R$. If $R = 0$, we mean that series converges only when $x = a$ and if $R = \infty$, then we mean the series converges for all $x \in \mathbb{R}$. The *interval of convergence* is the interval of x values such that the power series converges.

2.1 Basic Convergence Results

Since we are adding up a lot of terms, we need the terms to eventually be small to have any hope of the infinite sum converging.

Theorem 2. *If* $\lim_{n\to\infty} a_n \neq 0$ *or* $\lim_{n\to\infty} a_n$ *does not exist, then* $\sum_{n=0}^{\infty} a_n$ *diverges.*

However, if $\lim_{n\to\infty} a_n = 0$, then it does not automatically guarantee that the corresponding series converges. We need the terms $a_n \to 0$ fast enough for a series to converge.

Theorem 3. *If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $\lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = 0$.

2.2 Example Problems

We can use some algebra tricks to compute the exact value of certain infinite series.

Problem 1. ($\star$) Find

$$\sum_{n=1}^{\infty} 2^n \cdot 3^{1-n}.$$

Solution 1. Using algebra, we see that

$$\sum_{n=1}^{\infty} 2^n \cdot 3^{1-n} = 3\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 3 \cdot \frac{2}{3} \cdot \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 3 \cdot \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n.$$

Therefore, the formula for the geometric series implies that

$$\sum_{n=1}^{\infty} 2^n \cdot 3^{1-n} = 3 \cdot \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{1-\frac{2}{3}} = 6.$$

Problem 2. ($\star\star$) Find

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Solution 2. Using partial fractions, we have

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

The mth partial sums form a telescoping series,

$$s_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \cdots - \frac{1}{m+1} = 1 - \frac{1}{m+1}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n(n+1)} = \lim_{m\to\infty} 1 - \frac{1}{m+1} = 1.$$

Problem 3. ($\star$) Find

$$\sum_{n=1}^{\infty} \sqrt[n]{n}.$$

Solution 3. For all $n \geq 1$, the function $f(x) = x^{\frac{1}{n}}$ is increasing, which means that

$$n \geq 1 \implies n^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1.$$

In particular, $n^{\frac{1}{n}} \geq 1$ for all $n \geq 1$. Therefore, the summands do not go to 0 and the series diverges,

$$\sum_{n=1}^{\infty} \sqrt[n]{n} = \infty.$$

Problem 4. $(\star\star\star)$ Let $p \in (0,1)$. Find

$$p\sum_{k=1}^{\infty} k(1-p)^{k-1}.$$

Solution 4. Since $k = \sum_{j=1}^{k} 1$, we can write the series as a double sum and interchange the order of summation

$$p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p\sum_{k=1}^{\infty}\sum_{j=1}^{k}(1-p)^{k-1} = p\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}(1-p)^{k-1}.$$

The formula for the sum of a geometric series implies that

$$\sum_{k=1}^{\infty}(1-p)^{k} = (1-p)\sum_{k=1}^{\infty}(1-p)^{k-1} = \frac{(1-p)}{1-(1-p)} = \frac{(1-p)}{p}. \tag{1}$$

Using this to compute our sum, we have

$$\begin{aligned}
p\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}(1-p)^{k-1} &= p\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(1-p)^{j+k-2} \\
&= p\sum_{j=1}^{\infty}(1-p)^{j-2}\sum_{k=1}^{\infty}(1-p)^{k} \\
&= p\sum_{j=1}^{\infty}\frac{(1-p)^{j-1}}{p} && \text{Geometric Series (1)} \\
&= \frac{1}{(1-p)}\sum_{j=1}^{\infty}(1-p)^{j} \\
&= \frac{1}{p}. && \text{Geometric Series (1)}
\end{aligned}$$