1 Properties of the Heat Equation on $\mathbb{R}$

Recall that the solution to

$$\begin{cases} u_t - ku_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} f(y,s)\,dyds. \tag{2}$$

1.1 Well-Posed

Given some minor integrability assumptions on g (bounded and continuous), we can prove the existence of a C^∞ solution to (1) using (2). We can show the solutions are also unique and stable.

Proposition 1 (*Uniqueness of Solutions that Decay at Infinity*)

If u and its derivatives decay at infinity, then (1) has a unique solution.

Proof. We use an energy argument.

Difference of Solutions: Suppose u_1 and u_2 are solutions to (1) that decay at infinity. By linearity, $v = u_1 - u_2$ solves

$$\begin{cases} u_t - ku_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = 0 & x \in \mathbb{R}. \end{cases} \tag{3}$$

To prove uniqueness, it suffices to show that $v \equiv 0$ on the domain of the solution.

Show the Energy is Zero: We consider the energy of the solution v to (3),

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} v^2\,dx.$$

By the assumptions on the decay of u, we can differentiate under the integral sign with respect to t to conclude that

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} v_t v\,dx \\ &= k\int_{-\infty}^{\infty} v_{xx} v\,dx && v_t - kv_{xx} = 0 \\ &= -k\int_{-\infty}^{\infty} v_x^2\,dx + (v_x v)\Big|_{x=-\infty}^{x=\infty} && \text{Integrate by Parts} \\ &= -k\int_{-\infty}^{\infty} v_x^2\,dx && \lim_{x\to\pm\infty} u = 0,\ \lim_{x\to\pm\infty} u_x = 0 \\ &\le 0. \end{aligned}$$

Since $E'(t) \le 0$, we can conclude that $E(t)$ is decreasing by the mean value theorem. Furthermore, the initial conditions imply

$$E(0) = \frac{1}{2}\int_{-\infty}^{\infty} v(x,0)^2\,dx = 0$$

because $v(x,0) = 0$. This implies that $E(t) \le 0$. Combined with the fact $E(t) \ge 0$ since it is the integral of non-negative functions, this implies

$$0 \le E(t) \le 0 \implies E(t) = 0 \quad \text{for all } t.$$

Show the Difference is Zero: Since $E(t)$ is the integral of a sum of squares of continuous functions, each term in the integrand must be 0 so

$$v^2(x,t) = 0 \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0 \implies v(x,t) \equiv 0.$$

Therefore, $u_1 = u_2$, so the solution to (3) is unique. $\square$

Remark 1. We can also prove uniqueness for the homogeneous heat equation using by applying the maximum principle covered in the next section and taking limits. We need to assume some integrability on g to ensure that this limiting procedure is valid.

Proposition 2 (*Stability of Homogeneous Solutions that Decay at Infinity*)

If $f = 0$ and u and its derivatives decay at infinity, then (1) is stable.

Proof. Let u_1 be the solution to the homogeneous version of (1) with initial data g_1 and u_2 be the solution to the homogeneous version of (1) with and initial data g_2. We consider the energy of the solution v to (3),

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} v^2 \, dx.$$

The computations in the proof of uniqueness imply that $E'(t) \leq 0$, so $E(t)$ is decreasing. Furthermore, we have

$$E(0) = \frac{1}{2}\int_{-\infty}^{\infty} v(x,0)^2 \, dx = \frac{1}{2}\int_{-\infty}^{\infty} (g_1(x) - g_2(x))^2 \, dx = 0$$

because $v_1(x,0) = g_1(x)$ and $v_2(x,0) = g_2(x)$. We define the L^2 norm of f as

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} f^2(x)\, dx\right).$$

Therefore, $E(t) \leq E(0) = \frac{1}{2}\|g_1 - g_2\|_2^2$ by the mean value theorem. For every $\epsilon > 0$, if we take $\|g_1 - g_2\|_2 \leq \epsilon$, then

$$\frac{1}{2}\|u_1 - u_2\|_2^2 = E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_1 - u_2)^2 \, dx \leq \frac{1}{2}\|g_1 - g_2\|_2^2 \implies \|u_1 - u_2\|_2 \leq \epsilon$$

for all t. This implies stability for all t in terms of the "square error". $\square$

Remark 2. The heat equation is not well-posed for $t < 0$. For example, take $u_n = \frac{1}{n}\sin(nx)e^{-n^2 kt}$.

1.2 Symmetry

It is easy to check (2) implies that the solution $u(x,t)$ inherits the symmetry properties of the initial conditions and inhomogeneous term,

Proposition 3 (*Symmetry*)

Let $u(x,t)$ be the solution to (1).

(i) If f and g are even in x then $u(x,t)$ is even in x.

(ii) If f and g are odd in x then $u(x,t)$ is odd in x.

This means we can use odd or even reflections to solve the heat equation on the half line, in exactly the same way as for the half line wave equation.

1.3 Example Problems

Problem 1.1. ($\star$) Solve the following IBVP

$$\begin{cases} u_t - ku_{xx} = 0 & x > 0,\ t > 0, \\ u|_{t=0} = g(x) & x > 0 \\ u|_{x=0} = 0 & t > 0. \end{cases}$$

Solution 1.1. Since we have Dirichlet boundary conditions, we can find a solution using an odd-extension. Define

$$g_{odd}(x) = \begin{cases} g(x) & x > 0 \\ 0 & x = 0 \\ -g(-x) & x < 0. \end{cases}$$

For $x > 0$, the particular solution is given by

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g_{odd}(y)\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)\, dy - \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} g(-y)\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{\infty}^{0} e^{-\frac{(x+\tilde{y})^2}{4kt}} g(\tilde{y})\, d\tilde{y} && \tilde{y} = -y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) g(y)\, dy. \end{aligned}$$

Problem 1.2. ($\star$) Solve the following IBVP

$$\begin{cases} u_t - ku_{xx} = 0 & x > 0,\ t > 0, \\ u|_{t=0} = g(x) & x > 0 \\ u_x|_{x=0} = 0 & t > 0. \end{cases}$$

Solution 1.2. Since we have Neumann boundary conditions, we can find a solution using an even-extension. Define

$$g_{even}(x) = \begin{cases} g(x) & x \geq 0 \\ g(-x) & x \leq 0. \end{cases}$$

For $x > 0$, the particular solution is given by

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g_{even}(y)\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)\, dy + \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} g(-y)\, dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)\, dy - \frac{1}{\sqrt{4\pi kt}} \int_{\infty}^{0} e^{-\frac{(x+\tilde{y})^2}{4kt}} g(\tilde{y})\, d\tilde{y} && \tilde{y} = -y \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right) g(y)\, dy. \end{aligned}$$

Problem 1.3. ($\star\star$) Solve the following IBVP

$$\begin{cases} u_t - ku_{xx} = 0 & x > 0,\ t > 0, \\ u|_{t=0} = 0 & x > 0 \\ u|_{x=0} = p(t) & t > 0. \end{cases}$$

Solution 1.3. We reduce to a problem with homogeneous boundary conditions by doing a change of variables.

Change of Variables: We define $v(x,t) = u(x,t) - p(t)$. It is easy to check that v solves

$$\begin{cases} v_t - kv_{xx} = -p'(t) & x > 0,\ t > 0, \\ v|_{t=0} = -p(0) & x > 0 \\ v|_{x=0} = 0 & t > 0. \end{cases}$$

Particular Solution: This is an inhomogeneous heat equation with Dirichlet boundary conditions, so we can solve this using an odd reflection. We define

$$g_{odd}(x) = \begin{cases} -p(0) & x > 0 \\ 0 & x = 0 \\ p(0) & x < 0. \end{cases} \quad \text{and} \quad f_{odd}(x,t) = \begin{cases} -p'(t) & x > 0 \\ 0 & x = 0 \\ p'(t) & x < 0. \end{cases}$$

By (2),

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g_{odd}(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} f_{odd}(y,s)\, dyds.$$

Proceeding like in Problem 1.1, this simplifies to

$$-\frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) p(0)\, dy - \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left(e^{-\frac{(x-y)^2}{4k(t-s)}} - e^{-\frac{(x+y)^2}{4k(t-s)}} \right) p'(s) dyds. \tag{4}$$

Original Solution: We now write our solution in terms of u to conclude

$$u(x,t) = v(x,t) + p(t),$$

where $v(t)$ was defined in (4).

Remark 3. To solve an inhomogeneous Neumann problem $u_x|_{x=0} = p(t)$, you do the change of variables $v(x,t) = u(x,t) - xp(t)$ to reduce it to the homogeneous Neumann problem.

2 Properties of the Heat Equation on Finite Regions

2.1 Maximum Principle

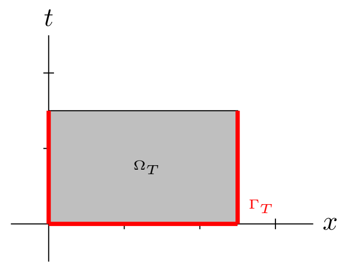
Consider the closed rectangular domain

$$\Omega_T = \{(x,t) : 0 < x < L, 0 < t \leq T\}.$$

We define the parabolic boundary of Ω_T by

$$\Gamma_T = \{(x,t) : x \in [0,L], t = 0 \text{ or } x = 0, t \in [0,T] \text{ or } x = L, t \in [0,T]\}$$

to be boundary of Ω_T without the top line $\{0 < x < L\} \times \{t = T\}$.



Remark 4. The restriction that $x \in [0,L]$ is not important. The maximum principle will hold for any finite interval $x \in [a,b]$ by applying the maximum principle to $v(x,t) = u(x-a,t)$.

If u is continuous on $\bar{\Omega}_T$ and satisfies the heat equation $u_t - ku_{xx} = 0$ on Ω_T, then the maximum value of u occurs on Γ_T. That is, the maximum of u is determined by the initial and boundary conditions.

Theorem 1 (*The Maximum Principle*)

If u is continuous and satisfies $u_t - ku_{xx} = 0$ on Ω_T, then

$$\max_{\bar{\Omega}_T} u = \max_{\Gamma_T} u. \tag{5}$$

Proof. Since u is continuous and $\bar{\Omega}_T$ is compact, u attains a global maximum value at some $(x_0,t_0) \in \bar{\Omega}_T$. We will show that (x_0,t_0) must be on Γ_T.

Perturbing the Solution: We will add a perturbation to the u to force its second derivative in x to have a sign. For $\epsilon > 0$, consider

$$v(x,t) = u(x,t) + \epsilon x^2.$$

Notice that v satisfies the *diffusion inequality* on Ω_T

$$v_t - kv_{xx} = u_t - ku_{xx} - 2\epsilon k = -2\epsilon k < 0. \tag{6}$$

We will use this fact to show that v cannot have an interior maximum.

Location of Maximum: The maximum must occur in one of 3 places.

1. **Interior:** Suppose that v attains its global maximum at (x_0,t_0) in the interior of the rectangle Ω_T. Since $v(x_0,t_0)$ is an interior maximum, the second derivative test implies that

$$v_t(x_0,t_0) = v_x(x_0,t_0) = 0 \quad \text{and} \quad v_{xx}(x_0,t_0) \leq 0 \implies v_t(x_0,t_0) - kv_{xx}(x_0,t_0) \geq 0.$$

   This contradicts the *diffusion inequality* (6), so v cannot attains its maximum in Ω_T.

2. **Top:** Similarly, suppose that v attains its global maximum at some point (x_0, T) on the top of the rectangle Ω_T. On the boundary, the second derivative test and the fact $v(x_0, T) \geq v(x_0, t)$ for any $t < T$, implies that

$$v_t(x_0, T) \geq 0,\ v_x(x_0, T) = 0 \quad \text{and} \quad v_{xx}(x_0, T) \leq 0 \implies v_t(x_0, T) - kv_{xx}(x_0, T) \geq 0.$$

   This contradicts the *diffusion inequality* (6), so v cannot attain the maximum on the top of Ω_T.

3. **Parabolic Boundary:** The extreme value theorem implies that v must attain its maximum somewhere, so it must attain its global maximum on Γ_T, i.e.

$$\max_{\bar{\Omega}_T} v = \max_{\Gamma_T} v.$$

Removing the Perturbation: We have shown that $v(x, t)$ attains a maximum at some point $(x_0, t_0) \in \Gamma_T$. Since $x \in [0, L]$ so $x^2 \leq L^2$ for all $(x, t) \in \bar{\Omega}_T$. Since $0 \leq \epsilon x^2 \leq \epsilon L^2$ on $\bar{\Omega}_T$,

$$\max_{\bar{\Omega}_T} u \leq \max_{\bar{\Omega}_T} v \leq v(x_0, t_0) \leq u(x_0, t_0) + \epsilon x_0^2 \leq \max_{\Gamma_T} u + \epsilon L^2.$$

The upperbound holds for all $\epsilon > 0$, so taking $\epsilon \to 0$ implies

$$\max_{\bar{\Omega}_T} u \leq \max_{\Gamma_T} u \implies \max_{\bar{\Omega}_T} u = \max_{\Gamma_T} u$$

since $\Gamma_T \subseteq \bar{\Omega}_T$. □

Remark 5. A minimum principle also hold by applying the maximum principle to $v = -u$. That is, if u is continuous and satisfies $u_t - ku_{xx} = 0$ on Ω_T, then

$$\min_{\bar{\Omega}_T} u = \min_{\Gamma_T} u. \tag{7}$$

Remark 6. It was essential that we have a strict inequality in (6). Without a strict inequality, an interior maximum might not lead to a contradiction. For example, the constant solution $u = 0$ is a continuous solution to the heat equation, but it has a global maximum on its interior.

2.2 Well-Posed

Consider the *Dirichlet problem for the heat equation*,

$$\begin{cases} u_t - ku_{xx} = f(x,t) & 0 < x < L,\ t > 0 \\ u|_{t=0} = g(x) & 0 < x < L \\ u|_{x=0} = p(t) & t > 0 \\ u|_{x=L} = q(t) & t > 0. \end{cases} \tag{8}$$

We will introduce techniques to solve this IBVP in Week 7. We can prove uniqueness and stability without even solving it.

Corollary 1 (*Uniqueness of the Inhomogeneous Dirichlet Problem*)

Continuous solutions to (8) are unique.

Proof. Uniqueness is a straightforward consequences of Theorem 1. We fix $T > 0$.

Uniqueness: Suppose u_1 and u_2 are solutions to (8). By linearity, $v = u_1 - u_2$ solves

$$\begin{cases} v_t - kv_{xx} = 0 & 0 < x < L,\ 0 < t \leq T \\ v|_{t=0} = 0 & 0 < x < L \\ v|_{x=0} = 0 & t > 0 \\ v|_{x=L} = 0 & t > 0. \end{cases} \tag{9}$$

Since v solves the heat equation and $v = 0$ on Γ_T, the maximum principle implies for any $(x,t) \in \bar{\Omega}_T$,

$$v(x,t) \leq \max_{\bar{\Omega}_T} v = \max_{\Gamma_T} v = 0.$$

Similarly, the minimum principle implies

$$v(x,t) \geq \min_{\bar{\Omega}_T} v = \min_{\Gamma_T} v = 0.$$

Therefore, $v(x,t) = 0$ on $\bar{\Omega}_T$, so $u_1 = u_2$ on $\bar{\Omega}_T$. Taking $T \to \infty$ implies uniqueness for all $t > 0$. □

Corollary 2 (*Stability of the Homogeneous Dirichlet Problem*)

If $f = 0$, then continuous solutions to (8) are stable.

Proof. Stability is a straightforward consequences of Theorem 1. Let u_1 be the solution to (8) with initial data (g_1, p_1, q_1) and u_2 be the solution to (8) with initial data (g_2, p_2, q_2). If we define $v = u_1 - u_2$, then v satisfies (8) with initial data $(g_1 - g_2, p_1 - p_2, q_1 - q_2)$. The maximum principle implies for any $(x,t) \in \bar{\Omega}_T$,

$$v = u_1 - u_2 \leq \max(\|g_1 - g_2\|_\infty, \|p_1 - p_2\|_\infty, \|q_1 - q_2\|_\infty),$$

and the minimum principle implies

$$v = u_1 - u_2 \geq -\max(\|g_1 - g_2\|_\infty, \|p_1 - p_2\|_\infty, \|q_1 - q_2\|_\infty).$$

For every $\epsilon > 0$, if we take $\max(\|g_1 - g_2\|_\infty, \|p_1 - p_2\|_\infty, \|q_1 - q_2\|_\infty) \leq \epsilon$, then

$$\|u_1 - u_2\|_T \leq \epsilon.$$

□

2.3 Example Problems

Problem 2.1. ($\star$) Let $u(x,t) = 1 - x^2 - 2t$ and $\Omega_T = \{0 \leq x \leq 1, 0 \leq t \leq T\}$. Find the maximum and minimum values of u on Ω_T.

Solution 2.1. It is easy to see that

$$u_t - u_{xx} = -2 + 2 = 0$$

so u solves the heat equation on Ω_T. By the maximum and minimum principles, it suffices to look for minimizers and maximizers on Γ_T, the shaped boundary of the closed rectangle.

We optimize over each of the three sides of the rectangle.

1. $\{0 \leq x \leq 1, t = 0\}$: We have
$$u(x,0) = 1 - x^2 \quad 0 \leq x \leq 1,$$
has a maximum of 1 when $x = 0$ and a minimum of 0 when $x = 1$.

2. $\{x = 0, 0 \leq t \leq T\}$: We have
$$u(0,t) = 1 - 2t \quad 0 \leq t \leq T,$$
has a maximum of 1 when $t = 0$ and a minimum of $1 - 2T$ when $t = T$.

3. $\{x = 1, 0 \leq t \leq T\}$: We have
$$u(1,t) = -2t \quad 0 \leq t \leq T,$$
has a maximum of 0 when $t = 0$ and a minimum of $-2T$ when $t = T$.

From above, we see that the maximum of 1 occurs at the point $x = 0, t = 0$ and the minimum of $-2T$ occurs when $x = 1, t = T$.