1 The Heat Equation on $\mathbb{R}$

The one dimensional *heat equation* models the temperature in a rod.

Definition 1. For parameter $k \in \mathbb{R}^+$, the homogeneous *heat equation* on $\mathbb{R} \times \mathbb{R}^+$ is

$$u_t - ku_{xx} = 0. \tag{1}$$

The corresponding IVP for the inhomogeneous heat equation is

$$\begin{cases} u_t - ku_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}. \end{cases} \tag{2}$$

The solution to this equation is derived using the method of *self similar solutions.*

Theorem 1 (*Solution to the Heat Equation*)

(a) The fundamental solution to (1) is

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}. \tag{3}$$

(b) The particular solution to (2) is given by,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} G(x-y,t)g(y)\,dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)f(y,s)\,dyds \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} g(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} f(y,s)\,dyds. \end{aligned} \tag{4}$$

Remark 1. The general solution does not have simple form in this problem. Instead, we will find the *fundamental solution* which is defined as the derivative in x of the solution to the problem

$$\begin{cases} u_t - ku_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = \mathbb{1}_{[0,\infty)}(x) & x \in \mathbb{R}. \end{cases} \tag{5}$$

By linearity, the solution to (2) can be pieced together from the fundamental solution.

Remark 2. We can think of (4) as the analogue of the solution for linear odes. Consider the ODE

$$y' - y = f(t), \qquad y(0) = g.$$

The solution when $f(t) = 0$ is solved by the integrating factor e^t, giving

$$y(t) = \exp(t) \cdot g + \int_0^t \exp(t-s) \cdot f(s)\,ds.$$

It turns out that (2) can be solved using operators on function spaces. The corresponding solution (4) can be written in terms of convolutions as

$$u(x,t) = G(x,t) * g(x) + \int_0^t G(x,t-s) * f(x,s)\,ds.$$

This operator approach to PDEs and the similarity between $G(x,t)$ and the exponential function is a topic called *semigroup theory*, which appears when we treat PDEs as an ODE for Banach space valued functions. This theory allows us to use ODEs to study stochastic differential equations.

1.1 Derivation of the Fundamental Solution

We will reduce the heat equation into a single variable problem by looking for a self similar solution. We will first solve (2) for a special choice of initial condition that will allow us to recover more general initial conditions using linearity. We want to find a solution v to the IVP,

$$\begin{cases} v_t - kv_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ v|_{t=0} = \mathbb{1}_{[0,\infty)}(x) & x \in \mathbb{R}. \end{cases} \tag{6}$$

Restriction to Self Similar Solutions: Our goal is to find α, β so that the solution v of the inhomogeneous heat equation is invariant under the *dilation scaling*, that is

$$v(x,t) = \lambda^\alpha v(\lambda^\beta x, \lambda t) \tag{7}$$

for all $\lambda > 0$, $x \in \mathbb{R}$ and $t > 0$. To simplify notation, we define $w(x,t;\lambda) := \lambda^\alpha v(\lambda^\beta x, \lambda t)$. To find such α and β, we plug w into the left hand sides of the equations in (6) to see that

$$w_t - kw_{xx} = \lambda^{\alpha+1} v_t(\lambda^\beta x, \lambda t) - k\lambda^{\alpha+2\beta} v_{xx}(\lambda^\beta x, \lambda t)$$

where v_x and v_t denote the partial derivatives with respect to the first and second coordinates of v, and

$$w|_{t=0} = \lambda^\alpha \mathbb{1}_{[0,\infty)}(\lambda^\beta x) = \lambda^\alpha \mathbb{1}_{[0,\infty)}(x)$$

because for $\lambda > 0$, $\lambda^\beta x \geq 0$ if and only if $x \geq 0$. If we take $\alpha = 0$ and $\beta = \frac{1}{2}$, then the computations above implies w also solves (6) for all λ. Since both v and w solve the same IVP, we can take solutions such that $v = w$, which implies (7) holds for

$$v(x,t) = v(\lambda^{\frac{1}{2}} x, \lambda t) \qquad \text{for all} \qquad \lambda > 0,\ x \in \mathbb{R},\ t > 0. \tag{8}$$

Reduction to an ODE: Since (8) holds for all $\lambda > 0$, we can define $\lambda = \frac{1}{t}$ to conclude that

$$v(x,t) = v(t^{-\frac{1}{2}} x, 1) = V(t^{-\frac{1}{2}} x) \qquad \text{where} \qquad V(s) = v(s,1) \tag{9}$$

solves the heat equation (1). Notice that V is a single variable function, so we can now solve for V using ODEs. Notice that

$$v_t - kv_{xx} = -\frac{x}{2t^{\frac{3}{2}}} V'(t^{-\frac{1}{2}} x) - \frac{k}{t} V''(t^{-\frac{1}{2}} x) = 0 \implies -\frac{x}{2t^{\frac{1}{2}}} V'(t^{-\frac{1}{2}} x) - kV''(t^{-\frac{1}{2}} x) = 0.$$

If we define $z = t^{-\frac{1}{2}} x$, then the above equation simplifies to

$$kV''(z) + \frac{z}{2} V'(z) = 0.$$

Finding the General Solution: If we define $\tilde{G}(z) = V'(z)$, then we have the first order equation

$$\tilde{G}'(z) + \frac{z}{2k} \tilde{G}(z) = 0 \implies \tilde{G}(z) = V'(z) = Ce^{-\frac{z^2}{4k}}$$

after using an integrating factor of the form $I(z) = e^{\frac{z^2}{4k}}$. Integrating this implies that

$$V(z) = C \int_0^z e^{-\frac{s^2}{4k}}\, ds + D \implies v(x,t) = V\Big(\frac{x}{\sqrt{t}}\Big) = C \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4k}}\, ds + D$$

for some unknown coefficients C and D.

Finding the Particular Solution: We now plug our general solution to solve for C and D. Since

$$\lim_{t \to 0^+} \frac{x}{\sqrt{t}} = \begin{cases} \infty & x > 0 \\ -\infty & x < 0, \end{cases}$$

and $\int_0^\infty e^{-\frac{s^2}{4k}}\,ds = \sqrt{k\pi}$, we see that

$$\lim_{t\to 0^+} V\Big(\frac{x}{\sqrt{t}}\Big) = \begin{cases} C\sqrt{k\pi} + D & x > 0 \\ -C\sqrt{k\pi} + D & x < 0 \end{cases} = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

provided that $C = \frac{1}{\sqrt{4k\pi}}$ and $D = \frac{1}{2}$. Therefore, the particular solution to (5) is given by

$$v(x,t) = V\Big(\frac{x}{\sqrt{t}}\Big) = \frac{1}{\sqrt{4k\pi}}\int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4k}}\,ds + \frac{1}{2} = \frac{1}{2}\mathrm{erf}\Big(\frac{x}{\sqrt{4kt}}\Big) + \frac{1}{2} \tag{10}$$

after a change of variables.

Finding the Fundamental Solution: The fundamental solution is defined to be the partial derivative if v with respect to x, so

$$G(x,t) = \frac{\partial}{\partial x}V\Big(\frac{x}{\sqrt{t}}\Big) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}. \tag{11}$$

It is easy to check that $G(x,t)$ also solves the heat equation because if v solves the heat equation, then v_x also solves the heat equation.

1.2 Derivation of the Particular Solution to the Homogeneous IVP

Suppose that g is a smooth function that vanishes at $\pm\infty$, i.e. $g(\pm\infty) = 0$. We want to use the fundamental solution to find solve the homogeneous version of (2)

$$\begin{cases} u_t - ku_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}. \end{cases} \tag{12}$$

Notice that we can write $g(x)$ as an integral against the step function,

$$g(x) = \int_{-\infty}^{x} g'(y)\,dy = \int_{-\infty}^{\infty} \mathbb{1}_{(-\infty,x]}(y)g'(y)\,dy = \int_{-\infty}^{\infty} \mathbb{1}_{[0,\infty)}(x-y)g'(y)\,dy \tag{13}$$

since $y \le x$ if and only if $x - y \ge 0$. Because of this fact, we can use the fact that v defined in (10) solves (6) to derive a solution for general initial conditions. We will show

$$u(x,t) = \int_{-\infty}^{\infty} v(x-y,t)g'(y)\,dy = -\int_{-\infty}^{\infty} \partial_y v(x-y,t)g(y)\,dy = \int_{-\infty}^{\infty} G(x-y,t)g(y)\,dy \tag{14}$$

solves (12). We integrated by parts and used the fact g vanishes at $\pm\infty$ to get the second equality, and we used the definition of G defined in (11) to get the last equality. We now check that u solves both the PDE and the initial condition.

1. *PDE:* Assuming that the integral is well defined, we can pass the derivatives inside to see that

$$u_t - ku_{xx} = \int_{-\infty}^{\infty} \underbrace{(\partial_t G(x-y,t) - k\partial_x^2 G(x-y,t))}_{=0} g(y)\,dy = 0,$$

   since G solves the heat equation and a translation of $G(x-y,t)$ solves the heat equation for all fixed y.

2. *Initial Condition:* Since v solves (6), $\lim_{t\to 0^+} v(x,t) = \mathbb{1}_{[0,\infty)}(x)$, so (13) implies

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+}\int_{-\infty}^{\infty} v(x-y,t)g'(y)\,dy = \int_{-\infty}^{\infty} \mathbb{1}_{[0,\infty)}(x-y)g'(y)\,dy = g(x). \tag{15}$$

Therefore, (14) is a solution to (12).

Remark 3. With a bit of work, one can remove the conditions of smoothness and decay at infinity on g. We essentially only need the integral to be finite.

1.3 Derivation of the Particular Solution to the Inhomogeneous IVP

We will derive the solution to the inhomogeneous heat equation using the solution to the homogeneous IVP and a general result called *Duhamel's principle*. By linearity, it suffices to solve

$$\begin{cases} u_t - ku_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = 0 & x \in \mathbb{R}. \end{cases} \tag{16}$$

Duhamel's principle states that the solution must be of the form

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) f(y,s)\, dyds. \tag{17}$$

We will verify that this solution satisfies (16) under the additional assumption that f is smooth, bounded, and decays at $\pm\infty$. These conditions can be relaxed, but the proof becomes more technical.

1. *PDE:* We differentiate u with respect to t using Leibniz's rule to see that

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t} \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) f(y,s)\, dyds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(x-y,t-s) f(y,s)\, dyds + \lim_{s\to t} \int_{-\infty}^{\infty} G(x-y,t-s) f(y,s)\, dy. \end{aligned}$$

A limit appears because there is a singularity when $s = t$. The first term simplifies nicely because $G(x-y,t-s)$ is a solution to the heat equation, so

$$\int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(x-y,t-s) f(y,s)\, dyds = k \int_0^t \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} G(x-y,t-s) f(y,s)\, dyds = ku_{xx}. \tag{18}$$

To compute the second term, we integrate by parts and use the fact f vanishes at $\pm\infty$ to see,

$$\begin{aligned} \lim_{s\to t} \int_{-\infty}^{\infty} G(x-y,t-s) f(y,s)\, dy &= \lim_{s\to t} \int_{-\infty}^{\infty} v(x-y,t-s) f_y(y,s)\, dy && \text{See equation (14)} \\ &= \int_{-\infty}^{\infty} \mathbb{1}_{[0,\infty)}(x-y) f_y(y,t)\, dy && \text{See equation (15)} \\ &= f(x,t). && \text{See equation (13)} \end{aligned}$$

2. *Initial Condition:* The outer integral is in t and f is assumed to be bounded so

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s) f(y,s)\, dyds = 0,$$

for every $x \in \mathbb{R}$ by Hölder's inequality.

Remark 4. The fundamental solution actually corresponds to the heat equation with a Dirac δ-function as the initial condition. The "function" $\delta(x)$ is called a distribution, which are a class of generalized functions in mathematics. It is the distributional derivative of the step function $\mathbb{1}_{[0,\infty)}(x)$. We define

$$\langle f, \delta \rangle = \int_{-\infty}^{\infty} f(y)\delta(y)\, dy := f(0). \tag{19}$$

If we define $\delta_a(x) = \delta(x-a)$, then this distribution means evaluation of a function at a after integration,

$$\langle f, \delta_a \rangle = \int_{-\infty}^{\infty} f(y)\delta(y-a)\, dy := f(a). \tag{20}$$

Using the δ function, we could have used the fact

$$\lim_{s\to t} G(x-y,t-s) = \delta_0(x-y) = \delta_y(x)$$

to verify the particular solution without integrating by parts.

1.4 Example Problems

Problem 1.1. ($\star$) Solve the following IVP

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = \mathbb{1}_{[a,\infty)}(x) & x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$ and

$$\mathbb{1}_{[a,\infty)}(x) = \begin{cases} 1 & x \geq a \\ 0 & x < a. \end{cases}$$

Solution 1.1. The particular solution of this IVP is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_a^\infty e^{-\frac{(y-x)^2}{4kt}}\, dy.$$

We can use a change of variables to write this function in terms of error functions. Using the change of variables

$$p = \frac{y-x}{\sqrt{4kt}}, \qquad \sqrt{4kt}\, dp = dy, \qquad y = a \implies p = \frac{a-x}{\sqrt{4kt}}, \qquad y = \infty \implies p = \infty$$

we get

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{a-x}{\sqrt{4kt}}}^\infty e^{-p^2}\, dp &= \frac{1}{\sqrt{\pi}} \int_{\frac{a-x}{\sqrt{4kt}}}^0 e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-p^2}\, dp \\ &= -\frac{1}{\sqrt{\pi}} \int_0^{\frac{a-x}{\sqrt{4kt}}} e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-p^2}\, dp \\ &= -\frac{1}{2}\operatorname{erf}\left(\frac{a-x}{\sqrt{4kt}}\right) + \frac{1}{2}. \end{aligned}$$

Remark 5. The particular solution of this IVP is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_a^\infty e^{-\frac{(y-x)^2}{4kt}}\, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{P}(X \geq a)$. This can be written in terms of the error function as

$$u(x,t) = \mathbb{P}(X \geq a) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{a-x}{\sqrt{4kt}}\right).$$

Problem 1.2. ($\star\star\star$) Solve the following IVP

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = \mathbb{1}_{[-1,1]}(x) & x \in \mathbb{R}, \end{cases}$$

where

$$\mathbb{1}_{[-1,1]}(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & x > 1. \end{cases}$$

Solution 1.2. We can use a change of variables to write this function in terms of error functions like in Problem 1.1. Instead, we can use linearity and our result in the last problem to simplify this problem. It is easy to check that

$$\mathbb{1}_{[-1,1]}(x) = \mathbb{1}_{[-1,\infty)}(x) - \mathbb{1}_{(1,\infty)}(x).$$

In particular, if we let u_1 be a solution to the heat equation with initial condition $\mathbb{1}_{[-1,\infty)}(x)$ and u_2 be a solution to the heat equation with initial condition $-\mathbb{1}_{(1,\infty)}(x)$, then it is easy to check that $u = u_1 + u_2$ is a solution to the PDE given in this problem. Therefore, by linearity and our result from Problem 1.1, we have

$$\begin{aligned} u(x,t) = u_1(x,t) + u_2(x,t) &= -\frac{1}{2}\mathrm{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right) + \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) - \frac{1}{2} \\ &= -\frac{1}{2}\mathrm{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right) + \frac{1}{2}\mathrm{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) \\ &= \frac{1}{2}\mathrm{erf}\left(\frac{1+x}{\sqrt{4kt}}\right) - \frac{1}{2}\mathrm{erf}\left(\frac{x-1}{\sqrt{4kt}}\right) \end{aligned}$$

since the error function is odd.

Problem 1.3. ($\star$) Solve the following IVP

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = e^{ax} & x \in \mathbb{R}, \end{cases}$$

where $a \in \mathbb{R}$.

Solution 1.3. The particular solution of this IVP is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay} \, dy.$$

Since the initial data is an exponential, we can complete the square to simplify the integral. Completing the square in y, we have

$$\begin{aligned} -\frac{(y-x)^2}{4kt} + ay &= -\frac{y^2 - 2xy + x^2 - 4ktay}{4kt} \\ &= -\frac{y^2 - (2x + 4kta)y + x^2}{4kt} \\ &= -\frac{(y - (x + 2kta))^2 - (x + 2kta)^2 + x^2}{4kt} \\ &= -\frac{(y - (x + 2kta))^2}{4kt} + ax + a^2kt. \end{aligned}$$

Therefore, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt} + ax + a^2kt} \, dy = e^{ax+a^2kt} \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt}} \, dy}_{=1} = e^{ax+a^2kt}.$$

See either Remark 6 or Remark 7 for the reasoning why the Gaussian integral is equal to 1.

Remark 6. In general, for any $a \in \mathbb{R}$ and $b > 0$ we have

$$\frac{\sqrt{b}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-b(y-a)^2}\, dy = 1.$$

To see this we can do the change of variables $p = \sqrt{b}(y-a)$ to see that

$$\frac{\sqrt{b}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-b(y-a)^2}\, dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\, dp = \frac{1}{2}\big(\operatorname{erf}(\infty) - \operatorname{erf}(-\infty)\big) = 1.$$

Remark 7. To see that the integral

$$\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+2kta))^2}{4kt}}\, dy = 1,$$

we can use the fact that this Gaussian integral corresponds to the p.d.f. of a Gaussian random variable with mean $(x + 2kta)$ and variance $2kt$, so its integral over $\mathbb{R}$ is equal to 1.

Remark 8. The particular solution of this PDE is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} e^{ay}\, dy.$$

If set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, e^{aX}$. In particular, using the formula for the moment generating function of the Gaussian random variable, we have

$$u(x,t) = \mathbb{E}\, e^{aX} = e^{a\mu + \frac{1}{2}a^2\sigma^2} = e^{ax + a^2kt}.$$

Problem 1.4. ($\star$) Solve the following IVP

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = x & x \in \mathbb{R}. \end{cases}$$

Solution 1.4. The particular solution of this IVP is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y\, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, X$. In particular, since the mean of this normal distribution is x, we have

$$u(x,t) = \mathbb{E}\, X = x.$$

Alternative Solution: We can add and subtract x and integrate to conclude

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} (y-x) e^{-\frac{(y-x)^2}{4kt}}\, dy + \frac{x}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}}\, dy = x,$$

since the first integral is 0 by symmetry and the second integral is equal to x, by Remark 6.

Problem 1.5. ($\star$) Solve the following IVP

$$\begin{cases} u_t = ku_{xx} & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = x^2 & x \in \mathbb{R}. \end{cases}$$

Solution 1.5. The particular solution of this IVP is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y^2 \, dy.$$

If we set $X \sim N(x, 2kt)$, then we have $u(x,t) = \mathbb{E}\, X^2$. In particular, since the variance of this normal distribution is $2kt$ and the mean is x, we have

$$u(x,t) = \mathbb{E}\, X^2 = \mathrm{Var}(X) + (\mathbb{E}\, X)^2 = 2kt + x^2,$$

where we used the fact $\mathrm{Var}(X) = \mathbb{E}\, X^2 - (\mathbb{E}\, X)^2$.

Alternative Solution: We can use the change of variables $p = \frac{y-x}{\sqrt{4kt}}$ to see that

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y^2 \, dy \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + \sqrt{4kt}p)^2 e^{-p^2} \, dp \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x^2 + 2x\sqrt{2kt}p + 4ktp^2) e^{-p^2} \, dp \\
&= x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp && \text{Remark 6 and Symmetry} \\
&= x^2 - \frac{2kt}{\sqrt{\pi}} p e^{-p^2} \Big|_{p=-\infty}^{p=\infty} + \frac{2kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \, dp && \text{IBP} \\
&= x^2 + 2kt. && \text{Remark 6}
\end{aligned}$$

Remark 9. In general, monomial initial conditions can be computed directly using a change of variable $p = \frac{y-x}{\sqrt{4kt}}$ and integrating by parts. To simplify notation, we can also use the Gaussian integration by parts formula

$$\mathbb{E}\, gF(g) = \mathbb{E}\, F'(g)$$

where $g \sim N(0,1)$.

Problem 1.6. ($\star$) Solve the following IVP

$$\begin{cases} u_t - ku_{xx} = x^2 & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = 0 & x \in \mathbb{R}. \end{cases}$$

Solution 1.6. The particular solution of this PDE is given by

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} y^2 \, dyds.$$

Using Problem 1.5 to compute the inner integral, we see that

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} y^2 \, dyds = \int_0^t 2k(t-s) + x^2 ds = kt^2 + tx^2.$$