1 Properties of the Wave Equation on $\mathbb{R}$

Recall that the solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

is given by d'Alembert's formula

$$u(x,t) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds. \tag{2}$$

We can use this formula to derive several nice properties satisfied by the solutions to (1).

1.1 Causality

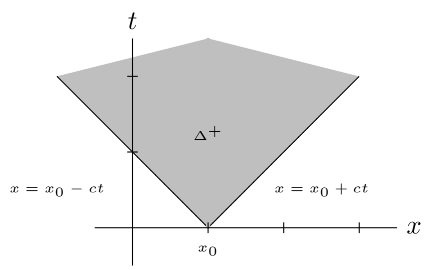
When $f \equiv 0$, d'Alembert's formula implies that waves propagate at speed c.

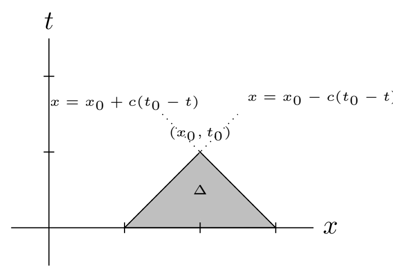
1. Domain of Influence: If $f \equiv 0$, $g(x_0)$ and $h(x_0)$ only affect the solution to (1) on the sector

$$\triangle^+(x_0, 0) = \{(x,t) : x_0 - ct \le x \le x_0 + ct\}.$$

2. Domain of Dependence: The solution to (1) at (x_0, t_0) only depends on f, g, h on the triangle

$$\triangle(x_0, t_0) = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : x_0 - c(t_0 - t) \le x \le x_0 + c(t_0 - t)\}.$$





Proposition 1 (*Causality*)

Suppose $f(x,t) \equiv 0$.

1. *If $g(x)$ and $h(x)$ vanish for $|x| > R$, then $u(x,t) = 0$ for x, t such that $|x| > R + ct$.*

2. *If $g(x)$ and $h(x)$ vanish for $|x - x_0| < R$ then $u(x,t) = 0$ for all $x, t \in \triangle(x_0, \frac{R}{c})$.*

Proof. When $f \equiv 0$, the wave equation is determined by the characteristic lines $x - ct = C_1$ and $x + ct = C_2$. The domain of influence and domain of dependence are simply the regions separated by the characteristic lines through the points x_0 and t_0. Part (2) follows by revisting the derivation of the first two terms in d'Alembert's formula and part (1) is the "inverse" restatement of part (2). $\square$

Remark 1. In higher dimensions, the domains of influence and dependence are cones.

Remark 2. Proposition 1 can be proved independently of d'Alembert's formula using a local energy argument. This is important because it implies that the energy of solutions with compactly supported initial data is differentiable, without relying on d'Alembert's formula.

1.2 Well-Posed

If we assume some regularity properties on the initial conditions, then d'Alembert's formula implies there exists a C^2 solution to (1). In fact, with a bit of work, we can show that the solutions are also unique and stable.

Proposition 2 (*The Inhomogeneous Wave Equation is Well-Posed*)

If $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and $f \in C^0(\mathbb{R})$ then the IVP (1) is well-posed.

Proof. We check the three conditions that define a well-posed solution.

Existence: The existence of a solution with continuous second derivative follows immediately from d'Alembert's formula (2).

Uniqueness: We use an energy argument to prove uniqueness.

Difference of Solutions: Suppose u_1 and u_2 are C^2 solutions to (1). By linearity, $v = u_1 - u_2$ solves

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & x \in \mathbb{R}, t > 0 \\ v|_{t=0} = 0 & x \in \mathbb{R} \\ v_t|_{t=0} = 0 & x \in \mathbb{R}. \end{cases} \tag{3}$$

To prove uniqueness, it suffices to show that $v \equiv 0$ on the domain of the solution.

Show the Energy is Zero: We consider the energy of the solution v to (3),

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left(v_t^2 + c^2 v_x^2\right) dx.$$

Since the initial conditions are 0, $v(x,t) = 0$ for $|x| > 1 + ct$ by Proposition 1. By continuity, this implies the integrand and its derivative are both integrable, so we differentiate under the integral sign with respect to t to see

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} \left(v_t v_{tt} + c^2 v_x v_{xt}\right) dx && (4) \\ &= c^2 \int_{-\infty}^{\infty} \left(v_t v_{xx} + v_x v_{xt}\right) dx && v_{tt} - c^2 v_{xx} = 0 \quad (5) \\ &= c^2 \int_{-\infty}^{\infty} \left((v_x v_t)_x\right) dx && (v_x v_t)_x = v_{xx} v_t + v_x v_{xt} \quad (6) \\ &= \lim_{y \to \infty} \Big(v_x(y,t) v_t(y,t) - v_x(-y,t) v_t(-y,t)\Big) && (7) \\ &= 0. && (8) \end{aligned}$$

We used the fact the wave equation has finite propagation speed Proposition 1 (the initial conditions vanish outside for $|x| > 1$), so $\lim_{y \to \pm\infty} v_x(y,t) = 0$ for every fixed t. Since $E'(t) = 0$, we can conclude that $E(t)$ is constant by the mean value theorem. Furthermore, the initial conditions imply

$$E(0) = \frac{1}{2} \int_{-\infty}^{\infty} \left(v_t^2(x,0) + c^2 v_x^2(x,0)\right) dx = 0$$

because $v_t(x,0) = 0$ and $v(x,0) = 0 \implies v_x(x,0) = 0$. Combined with the fact $E(t)$ is constant, we have

$$E(t) = 0 \quad \text{for all } t.$$

Show the Difference is Zero: Since $E(t)$ is the integral of a sum of squares of continuous functions, each term in the integrand must be 0 by the vanishing theorem,

$$v_x^2(x,t) = 0, v_t^2(x,t) = 0 \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0 \implies \nabla v(x,t) \equiv 0.$$

This implies that $v(x,t)$ is constant on $\mathbb{R} \times \mathbb{R}^+$. Since $v(0,0) = 0$, we can conclude that $v \equiv 0$ on $\mathbb{R} \times \mathbb{R}^+$, so $u_1 = u_2$ and the solution to (3) is unique.

Stable: Consider the uniform norms on $\mathbb{R}$ and $\mathbb{R} \times [0,T]$,

$$\|w\|_\infty = \sup_{x\in\mathbb{R}} |w(x)| \quad \text{and} \quad \|w\|_T = \sup_{x\in\mathbb{R}, t\in[0,T]} |w(x,t)|.$$

Let u_1 be the solution to (1) with initial data (f_1, g_1, h_1) and u_2 be the solution to (1) with initial data (f_2, g_2, h_2). Since the unique solutions to (1) is given by (2) the triangle inequality and Jensen's inequality implies that

$$\|u_1 - u_2\|_T \leq \|g_1 - g_2\|_\infty + T\|h_1 - h_2\|_\infty + \frac{T^2}{2}\|f_1 - f_2\|_T.$$

where $2cT$ is the biggest length of the the interval $[x - ct, x + ct]$ and cT^2 is the maximum volume of $\Delta(x,t)$ for $x, t \in \mathbb{R} \times [0,T]$. For every $\epsilon > 0$, if we take

$$\|g_1 - g_2\|_\infty, \|h_1 - h_2\|_\infty, \|f_1 - f_2\|_T < \frac{\epsilon}{1 + T + T^2},$$

then

$$\|u_1 - u_2\|_T \leq \epsilon,$$

proving continuity of the solutions with respect to the initial conditions. □

Remark 3. A similar energy argument can be used to prove uniqueness for the half line problems with Dirichlet and Neumann boundary conditions as well.

1.3 Symmetry

It is easy to check (2) implies that the solution $u(x,t)$ inherits the symmetry properties of the initial conditions and inhomogeneous term,

Proposition 3 (*Symmetry*)

Let $u(x,t)$ be the solution to (1).

(i) If f, g and h are even in x then $u(x,t)$ is even in x.

(ii) If f, g and h are odd in x then $u(x,t)$ is odd in x.

This property is the key to the reflection method we see in the next section.

1.4 Example Problems

Problem 1.1.

(a) Prove that the derivative of an odd function is even, and the derivative of an even function is odd.

(b) Prove that any antiderivative of an odd function is even, and the antiderivative of an even function is odd provided it passes through the origin.

Solution 1.1. We are assuming the functions are defined on an interval of the form $[-M, M]$.

(Part a) Suppose that $\phi(x)$ is odd, that is,

$$\phi(x) = -\phi(-x).$$

Differentiating both sides and applying the chain rule, we see

$$\phi'(x) = \phi'(-x),$$

which implies that $\phi'(x)$ is even. Similarly, suppose that $\phi(x)$ is even, that is,

$$\phi(x) = \phi(-x).$$

Differentiating both sides and applying the chain rule, we see

$$\phi'(x) = -\phi'(-x),$$

which implies that $\phi'(x)$ is odd.

(Part b) Suppose that $\phi(x)$ is odd, that is.

$$\phi(x) = -\phi(-x).$$

Integrating both sides starting at 0, and using a change of variables $\tilde{t} = -t$, we see

$$\int_0^x \phi(t)\,dt = -\int_0^x \phi(-t)\,dt \implies \Phi(x) := \int_0^x \phi(t)\,dt = \int_0^{-x} \phi(\tilde{t})\,d\tilde{t} =: \Phi(-x),$$

which implies that the integral $\Phi(x) = \int_0^x \phi(t)\,dt$ is even. Similarly, suppose that $\phi(x)$ is even, that is,

$$\phi(x) = \phi(-x).$$

Integrating both sides starting at 0, and using a change of variables $\tilde{t} = -t$, we see

$$\int_0^x \phi(t)\,dt = \int_0^x \phi(-t)\,dt \implies \Phi(x) := \int_0^x \phi(t)\,dt = -\int_0^{-x} \phi(\tilde{t})\,d\tilde{t} =: -\Phi(-x),$$

which implies that the integral $\Phi(x) = \int_0^x \phi(t)\,dt$ is odd.

Remark 4. If ϕ is odd, then it doesn't matter what the lower bound of integration is. For example, if we chose to integrate starting at $a \neq 0$, then we have

$$\int_a^x \phi(t)\,dt = -\int_a^x \phi(-t)\,dt \implies \int_a^x \phi(t)\,dt = \int_{-a}^{-x} \phi(\tilde{t})\,d\tilde{t}$$

It may appear that the term on the right is not of the same form as the term on the left because the lower bounds of integration are different, but we can split the region of integration and use a change of variables to conclude that

$$\begin{aligned}
\int_{-a}^{-x} \phi(\tilde{t})\,d\tilde{t} &= \int_{-a}^{0} \phi(\tilde{t})\,d\tilde{t} + \int_0^{-x} \phi(\tilde{t})\,d\tilde{t} \\
&= -\int_a^0 \phi(-t)\,dt + \int_0^{-x} \phi(\tilde{t})\,d\tilde{t} && \tilde{t} = -t \\
&= \int_a^0 \phi(t)\,dt + \int_0^{-x} \phi(\tilde{t})\,d\tilde{t} = \int_a^{-x} \phi(t)\,dt && \phi(-t) = -\phi(t)
\end{aligned}$$

This means if we pick any function in the family of anti-derivatives of an odd function, then it is always even. In particular, the integration constant can be arbitrary.

For example, the integral of $\phi(x) = x$ is given by

$$\int x\,dx = \frac{x^2}{2} + C,$$

which is even for all choices of $C \in \mathbb{R}$.

Remark 5. If ϕ is even, it is essential that we chose the starting point of integration to be 0. This is because we require $\Phi(0) = 0$, which holds in general if the lower bound of integration is 0. This means if we pick any function in the family of anti-derivatives of an even function, then we need to choose our integration constant to ensure that $\Phi(0) = 0$.

For example, the integral of $\phi(x) = x^2$ is given by,

$$\int x^2\,dx = \frac{x^3}{3} + C,$$

which is odd only when $C = 0$.

Problem 1.2. ($\star\star\star$) Prove the vanishing theorem. That is, suppose that $f(x)$ is a continuous function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and

$$\int_a^b f(x)\,dx = 0.$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution 1.2. On the contrary, suppose that $f(x^*) > 0$ for some point $x^* \in [a, b]$. Then by continuity, we must also have that $f(x) > 0$ on some interval $[k, \ell] \subset [a, b]$. By the mean value theorem of integration, there exists a $c \in [k, \ell]$ such that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k).$$

Since we also have that $f(x) > 0$ for all $x \in [k, \ell]$, we must have $f(c) > 0$, which implies that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k) > 0.$$

Since $f(x) \geq 0$, by the monotonicity of integration, the conclusion above implies

$$\int_a^b f(x)\,dx \geq \int_k^\ell f(x)\,dx > 0,$$

which contradicts the fact that $\int_a^b f(x)\,dx = 0$. Therefore, we must have that $f(x) = 0$ for all $x \in [a, b]$.

Remark 6. A variant of this result was used in the energy argument.

2 Wave Equation with One Boundary Condition

2.1 The Reflection Method for the Half Line

Suppose we have the wave equation on the half line. We want to reduce this problem to a PDE on the entire line by finding an appropriate extension of the initial conditions and source terms that satisfies the given boundary conditions. The choice of the extension only depends on the boundary conditions:

1. Dirichlet ($u|_{x=0} = 0$): We take an odd extension of the initial conditions

$$\phi_{odd} = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases}.$$

2. Neumann ($u_x|_{x=0} = 0$): We take an even extension of the initial conditions

$$\phi_{even} = \begin{cases} \phi(x) & x \geq 0 \\ \phi(-x) & x \leq 0 \end{cases}.$$

This extension reduces the wave equation on the half line to the full line, so we can apply D'Alembert's formula

$$u_{ext}(x,t) = \frac{g_{ext}(x+ct) + g_{ext}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h_{ext}(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dyds. \quad (9)$$

where the initial conditions and source terms are replaced by the respective odd or even extensions in space. The symmetry properties in Proposition 3 and an uniqueness property Remark 3 imply the restriction of the extended solution is the unique solution to the PDE on the half line. The general formulas for the homogeneous Dirichlet and Neumann problems are derived in the example Problems 2.1 and 2.2.

Remark 7. If we do an odd extension, then $u_{odd}(x,t)$ is odd in x by Proposition 3, so $u_{odd}|_{x=0} = 0$ automatically because continuous odd functions must pass through the origin.

Remark 8. Similarly, if we do an even extension, then $u_{even}(x,t)$ is even in x by Proposition 3, so $(u_{even})_x|_{x=0} = 0$ automatically because the derivative of continuous even functions must pass through the origin (this follows because the derivative of an even function is odd).

2.2 General Boundary Conditions

The reflection method does not work for inhomogeneous boundary conditions. For these questions, it is easiest to start from the general solution

$$u(x,t) = \phi(x-ct) + \psi(x+ct) \qquad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}$$

and solve for ϕ and ψ by using the initial and boundary conditions.

Remark 9. This approach also works for the half line problems with Dirichlet or Neumann boundary conditions. We can check that this approach will give us the extended D'Alembert's formula in (9).

2.3 Example Problems

Problem 2.1. ($\star\star$) Solve the following IBVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < \infty \\ u_t|_{t=0} = h(x) & 0 < x < \infty \\ u|_{x=0} = 0 & t > 0 \end{cases}.$$

Solution 2.1. Since we have Dirichlet boundary conditions, we can find a solution using an odd-extension. Define

$$g_{odd}(x) = \begin{cases} g(x) & x > 0 \\ 0 & x = 0 \\ -g(-x) & x < 0 \end{cases} \quad \text{and} \quad h_{odd}(x) = \begin{cases} h(x) & x > 0 \\ 0 & x = 0 \\ -h(-x) & x < 0. \end{cases}$$

By (9), the general solution for $x > 0$ is given by

$$u(x,t) = \frac{g_{odd}(x+ct) + g_{odd}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h_{odd}(s)\,ds.$$

We now examine the cases depending on the sign of $x - ct$:

1. For $x - ct > 0$, we have $g_{odd}(x \pm ct) = g(x \pm ct)$ and $h_{odd}(x \pm ct) = h(x \pm ct)$ so the solution is given by
$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$$

2. For $x - ct < 0$, we have $g_{odd}(x-ct) = -g(ct-x)$ and for $x - ct < s < 0$, $h_{odd}(s) = -h(-s)$, so
$$\begin{aligned} u(x,t) &= \frac{g(x+ct) - g(ct-x)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} h(s)\,ds - \int_{x-ct}^0 h(-s)\,ds\right) \\ &= \frac{g(x+ct) - g(ct-x)}{2} + \frac{1}{2c}\left(\int_0^{x+ct} h(s)\,ds + \int_{ct-x}^0 h(\tilde{s})\,d\tilde{s}\right) \qquad \tilde{s} = -s \\ &= \frac{g(x+ct) - g(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} h(s)\,ds. \end{aligned}$$

3. For $x - ct = 0$, we have $g_{odd}(x-ct) = 0$, so
$$u(x,t) = \frac{g(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} h(s)\,ds.$$

Problem 2.2. ($\star\star$) Solve the following IBVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < \infty \\ u_t|_{t=0} = h(x) & 0 < x < \infty \\ u_x|_{x=0} = 0 & t > 0 \end{cases}.$$

Solution 2.2. Since we have Neumann boundary conditions, we can find a solution using an even-extension. Define

$$g_{even}(x) = \begin{cases} g(x) & x \geq 0 \\ g(-x) & x \leq 0 \end{cases} \quad \text{and} \quad h_{even}(x) = \begin{cases} h(x) & x \geq 0 \\ h(-x) & x \leq 0. \end{cases}$$

By (9), the general solution for $x > 0$ is given by

$$u(x,t) = \frac{g_{even}(x+ct) + g_{even}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h_{even}(s)\,ds.$$

We now examine the cases depending on the sign of $x - ct$:

1. For $x - ct \geq 0$, we have $g_{even}(x \pm ct) = g(x \pm ct)$ and $h_{even}(x \pm ct) = h(x \pm ct)$ so the solution is given by
$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$$

2. For $x - ct < 0$, we have $g_{even}(x - ct) = g(ct - x)$ and for $x - ct < s < 0$, $h_{even}(s) = h(-s)$, so

$$\begin{aligned} u(x,t) &= \frac{g(x+ct) + g(ct-x)}{2} + \frac{1}{2c}\bigg(\int_0^{x+ct} h(s)\,ds + \int_{x-ct}^{0} h(-s)\,ds\bigg) \\ &= \frac{g(x+ct) + g(ct-x)}{2} + \frac{1}{2c}\bigg(\int_0^{x+ct} h(s)\,ds - \int_{ct-x}^{0} h(\tilde{s})\,d\tilde{s}\bigg) \qquad \tilde{s} = -s \\ &= \frac{g(x+ct) + g(ct-x)}{2} + \frac{1}{2c}\bigg(\int_0^{x+ct} h(s)\,ds + \int_{0}^{ct-x} h(s)\,ds\bigg). \end{aligned}$$

Problem 2.3. ($\star$) Solve the following IBVP

$$\begin{cases} u_{tt} = 4u_{xx} & 0 < x < \infty, \quad t > 0 \\ u|_{t=0} = 1 & 0 < x < \infty, \\ u_t|_{t=0} = 0 & 0 < x < \infty \\ u|_{x=0} = 0 & t > 0 \end{cases}.$$

Solution 2.3. We want to solve the wave equation on the half line with Dirichlet boundary conditions. We can use an odd reflection to extend the initial condition,

$$g_{odd}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}, \quad h_{odd}(x) = 0.$$

The particular solution to the extended PDE is

$$u(x,t) = \frac{g_{odd}(x+2t) + g_{odd}(x-2t)}{2}.$$

We now examine the cases depending on the sign of $x - 2t$:

1. For $x - 2t > 0$, we have $g_{odd}(x \pm 2t) = 1$ so the solution is given by
$$u(x,t) = \frac{1+1}{2} = 1.$$

2. For $x - 2t < 0$, we have $g_{odd}(x - 2t) = -1$ while $g_{odd}(x + 2t) = 1$, so

$$u(x,t) = \frac{1-1}{2} = 0.$$

3. When $x - 2t = 0$, we have $g_{odd}(x - 2t) = 0$ so

$$u(x,t) = \frac{1}{2}.$$

In summary, for $x > 0$ and $t > 0$, the solution is given by

$$u(x,t) = \begin{cases} 1 & x > 2t \\ \frac{1}{2} & x = 2t \\ 0 & x < 2t \end{cases}.$$

From here, it is clear that there is a singularity at the line $x = 2t$. This is because the odd extension of g is not continuous at $x = 0$.

2.4 General Boundary Conditions

Problem 2.4. ($\star\star\star$) Solve the following IBVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < \infty \\ u_t|_{t=0} = cg'(x) & 0 < x < \infty \\ u_x + \alpha u|_{x=0} = 0 & t > 0 \end{cases}.$$

Solution 2.4. Recall that the general solution of $u_{tt} - c^2 u_{xx} = 0$ is given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \text{ for } x, t > 0,$$

for some yet to be determined functions ϕ and ψ. Using the initial and boundary conditions, we can recover the specific form of ϕ and ψ.

Initial Condition: For $x > 0$, the first initial condition implies that

$$\phi(x+ct) + \psi(x-ct)\big|_{t=0} = g(x) \implies \phi(s) + \psi(s) = g(s) \text{ for } s > 0 \tag{10}$$

and the second boundary condition initial that

$$c\phi'(x+ct) - c\psi'(x-ct)\big|_{t=0} = cg'(x) \implies \phi'(s) - \psi'(s) = g'(s) \text{ for } s > 0. \tag{11}$$

Differentiating (10) and adding it to (11) implies that

$$2\phi'(s) = 2g'(s) \implies \phi(s) = g(s) + C \text{ for } s > 0. \tag{12}$$

Substituting this into (10) implies

$$g(s) + C + \psi(s) = g(s) \implies \psi(s) = -C \text{ for } s > 0. \tag{13}$$

Boundary Condition: Since $x + ct > 0$ on the domain, it remains to find ψ for $s < 0$. For $t > 0$, the boundary condition implies

$$\phi'(x+ct) + \psi'(x-ct) + \alpha\phi(x+ct) + \alpha\psi(x-ct)\big|_{x=0} = \phi'(ct) + \psi'(-ct) + \alpha\phi(ct) + \alpha\psi(-ct) = 0.$$

Since $\phi(ct) = g(ct) + C$ for $ct > 0$ from our computations above,

$$g'(ct) + \psi'(-ct) + \alpha g(ct) + \alpha C + \alpha\psi(-ct) = 0 \overset{s=-ct}{\implies} \psi'(s) + \alpha\psi(s) = -g'(-s) - \alpha g(-s) - \alpha C,$$

where $s < 0$. This is a linear first order ODE, so we can solve it using an integrating factor $e^{\alpha s}$,

$$\frac{d}{ds}\Big(e^{\alpha s}\psi(s)\Big) = -e^{\alpha s}g'(-s) - \alpha e^{\alpha s}g(-s) - \alpha C e^{\alpha s} \tag{14}$$

$$\implies \qquad \psi(s) = e^{-\alpha s}\int_0^s -e^{\alpha r}g'(-r) - \alpha e^{\alpha r}g(-r)\,dr - C + De^{-\alpha s} \tag{15}$$

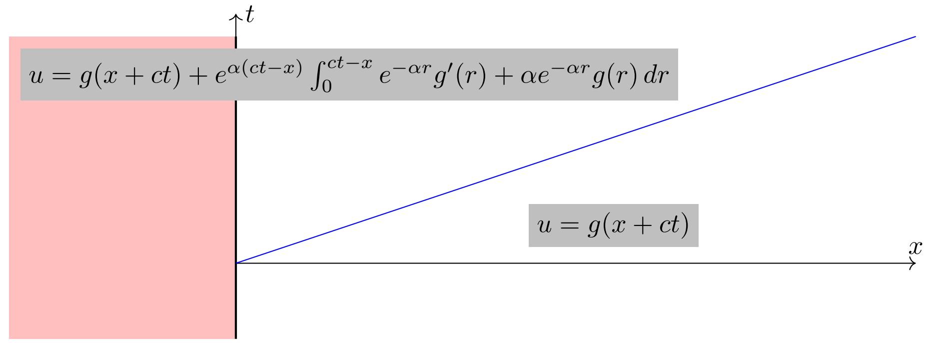
$$= e^{-\alpha s}\int_0^{-s} e^{-\alpha r}g'(r) + \alpha e^{-\alpha r}g(r)\,dr - C + De^{-\alpha s} \text{ for } s < 0. \tag{16}$$

Solution: For $x, t > 0$ the formulas in (12), (13) and (14) implies

$$u(x,t) = \begin{cases} g(x+ct) & x - ct > 0 \\ g(x+ct) + e^{\alpha(ct-x)}\displaystyle\int_0^{ct-x} e^{-\alpha r}g'(r) + \alpha e^{-\alpha r}g(r)\,dr + De^{\alpha(ct-x)} & x - ct < 0. \end{cases}$$

This formula is well defined because $g(s)$ is defined for $s > 0$. If we require our solution to be continuous, then as $x \to ct$ from the left we also require

$$g(2ct) = g(2ct) + D \implies D = 0.$$



Problem 2.5. ($\star\star$) Solve the following IBVP

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & x > -t, \quad t > 0 \\ u|_{t=0} = 0 & 0 < x < \infty \\ u_t|_{t=0} = 0 & 0 < x < \infty \\ u_x|_{x=-t} = \sin(t) & t > 0 \end{cases}.$$

Solution 2.5. Recall that the general solution of $u_{tt} - 4u_{xx} = 0$ is given by

$$u(x,t) = \phi(x+2t) + \psi(x-2t) \text{ for } x > -t,\ t > 0,$$

for some yet to be determined functions ϕ and ψ. Using the initial and boundary conditions, we can recover the specific form of ϕ and ψ.

Initial Condition: For $x > 0$, the first initial condition implies that

$$\phi(x+2t) + \psi(x-2t)\big|_{t=0} = 0 \implies \phi(s) + \psi(s) = 0 \text{ for } s > 0 \tag{17}$$

and the second boundary condition initial that

$$2\phi'(x+2t) - 2\psi'(x-2t)\big|_{t=0} = 0 \implies \phi'(s) - \psi'(s) = 0 \text{ for } s > 0. \tag{18}$$

Differentiating (17) and adding it to (18) implies that

$$2\phi'(s) = 0 \implies \phi(s) = C \text{ for } s > 0. \tag{19}$$

Substituting this into (17) implies

$$C + \psi(s) = 0 \implies \psi(s) = -C \text{ for } s > 0. \tag{20}$$

Boundary Condition: Since $x + 2t > 0$ It remains to find ψ for $s < 0$. For $t > 0$, the boundary condition implies

$$\phi'(x+2t) + \psi'(x-2t)\big|_{x=-t} = \phi'(t) + \psi'(-3t) = \sin(t).$$

Since $\phi(t) = C$ for $t > 0$ from our computations above,

$$\psi'(-3t) = \sin(t) \overset{s=-3t}{\implies} \psi'(s) = \sin\Big(-\frac{s}{3}\Big),$$

where $s < 0$. Integrating this, we see that

$$\psi(s) = 3\cos\Big(-\frac{s}{3}\Big) + D \text{ for } s < 0. \tag{21}$$

Solution: For $x > -t$, $t > 0$ the formulas in (19), (20) and (21) implies

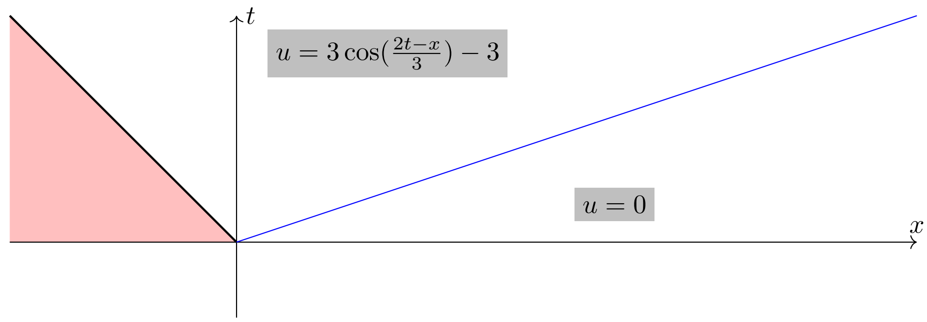
$$u(x,t) = \begin{cases} 0 & x > 2t \\ C + 3\cos(\frac{2t-x}{3}) + D & -t < x < 2t. \end{cases}$$

If we require our solution to be continuous, then as $x \to 2t$ from the left we also require

$$0 = C + 3\cos(0) + D \implies C + D = -3,$$

that is,

$$u(x,t) = \begin{cases} 0 & x > 2t \\ 3\cos(\frac{2t-x}{3}) - 3 & -t < x < 2t. \end{cases}$$



Remark 10. The odd and even reflection formulas in Problems 2.4 and 2.5 can also be derived directly from the general formulas using the approaches in Problems 2.1 and 2.2.