1 The Wave Equation on $\mathbb{R}$

The one dimensional *wave equation* models a vibrating string.

Definition 1. For parameter $c \in \mathbb{R}^+$, the homogeneous *wave equation* on $\mathbb{R} \times \mathbb{R}^+$ is

$$u_{tt} - c^2 u_{xx} = 0. \tag{1}$$

The corresponding IVP for the inhomogeneous wave equation is

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R}. \end{cases} \tag{2}$$

The solution to this equation is derived using the *method of characteristics.*

Theorem 1 (*Solution to the Wave Equation*)

(a) The general solution to (1) is

$$u(x,t) = \phi(x - ct) + \psi(x + ct), \tag{3}$$

where ϕ, ψ are arbitrary functions.

(b) The particular solution to (2) is given by d'Alembert's Formula,

$$u(x,t) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds. \tag{4}$$

1.1 Derivation of the General Solution

We give two derivations of the general solution (3).

1.1.1 Method 1: Factoring the Operator

We reduce the second order PDE to iterated first order PDEs and apply the methods from Week 2. We begin by factoring the linear operator $L[u] = (\partial_t^2 - c^2\partial_x^2)u$,

$$L[u] = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u.$$

Notice that if u is a solution to (1), then $L[u] = 0$. If we define $v = (\partial_t - c\partial_x)u = u_t - cu_x$, then

$$L[u] = 0 \iff (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = (\partial_t + c\partial_x)v = v_t + cv_x = 0.$$

This gives us the following system of first order equations

$$\begin{cases} u_t - cu_x = v \\ v_t + cv_x = 0 \end{cases}. \tag{5}$$

Solving the Second Equation: Using the general solution of the *transport equation* (see Week 2),

$$v_t + cv_x = 0 \implies v(x,t) = \varphi'(x - ct)$$

for some differentiable function φ' (this form was chosen to simplify notation).

Solving the First Equation: Since $v = u_t - cu_x$ to recover u, we need to solve

$$u_t - cu_x = \varphi'(x - ct).$$

This is a first order linear equation, so it suffices to solve the system

$$\frac{dt}{1} = \frac{dx}{-c} = \frac{du}{\varphi'(x - ct)}.$$

The equation involving the first and second terms gives us the characteristics

$$\frac{dt}{1} = \frac{dx}{-c} \implies x = -ct + C \implies C = x + ct.$$

Solving the equation involving the first third term implies

$$\frac{dt}{1} = \frac{du}{\varphi'(x - ct)} = \frac{du}{\varphi'(C - 2ct)} \implies u(x,t) = -\frac{1}{2c}\varphi(C - 2ct) + \psi(C) = -\frac{1}{2c}\varphi(x - ct) + \psi(x + ct).$$

If we define $\phi = -\frac{1}{2c}\varphi$, then we get the general solution

$$u(x,t) = \phi(x - ct) + \psi(x + ct).$$

1.1.2 Method 2: Change of Variables

We do a change of variables to simplify the form of the PDE. We begin by factoring the linear operator $L[u] = (\partial_t^2 - c^2\partial_x^2)u$,

$$L[u] = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u.$$

This factorization seems to suggest two characteristic curves

$$\frac{dt}{1} = \frac{dx}{c} \implies C = x - ct \quad \text{and} \quad \frac{dt}{1} = \frac{dx}{-c} \implies D = x + ct.$$

We will use these characteristics curves to define a change of variables that will greatly simplify the PDE. Consider the change of variables

$$\xi(x,t) = x - ct \quad \text{and} \quad \eta(x,t) = x + ct. \tag{6}$$

By the multivariable chain rule,

$$\partial_t u(\xi,\eta) = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = -cu_\xi + cu_\eta = (-c\partial_\xi + c\partial_\eta)u(\xi,\eta) \implies \partial_t = (-c\partial_\xi + c\partial_\eta)$$

and

$$\partial_x u(\xi,\eta) = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = u_\xi + u_\eta = (\partial_\xi + \partial_\eta)u(\xi,\eta) \implies \partial_x = (\partial_\xi + \partial_\eta).$$

In particular, these computations imply that the original operators are equal to

$$(\partial_t + c\partial_x) = ((-c\partial_\xi + c\partial_\eta) + c(\partial_\xi + \partial_\eta)) = 2c\partial_\eta$$

and

$$(\partial_t - c\partial_x) = ((-c\partial_\xi + c\partial_\eta) - c(\partial_\xi + \partial_\eta)) = -2c\partial_\xi.$$

Therefore, under the change of variables (6),

$$L[u] = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = (2c\partial_\eta)(-2c\partial_\xi)u = -4c^2u_{\xi\eta}.$$

If u satisfies (1), then $L[u] = 0$. Since $c \neq 0$, directly integrating this PDE (see Week 1) implies

$$0 = L[u] = -4c^2u_{\xi\eta} \implies u(\xi,\eta) = \phi(\xi) + \psi(\eta) \implies u(x,t) = \phi(x - ct) + \psi(x + ct),$$

after writing it back in the original variables using (6).

Remark 1. From the proofs, we see that the general solution (3) holds for $t < 0$ as well.

1.2 Derivation of the Particular Solution

We will now use the initial conditions (2) to find the particular form of the solution. By linearity, we can write the solution as $u = v + w$, where v solve the homoegenous IVP and w solves the inhomogeneous IVP with vanishing initial values,

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ v|_{t=0} = g(x) & x \in \mathbb{R}, \\ v_t|_{t=0} = h(x) & x \in \mathbb{R} \end{cases} \quad \text{and} \quad \begin{cases} w_{tt} - c^2 w_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ w|_{t=0} = 0 & x \in \mathbb{R}, \\ w_t|_{t=0} = 0 & x \in \mathbb{R}. \end{cases} \tag{7}$$

1.2.1 The Particular Solution for the Homogeneous IVP

Suppose that $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. We want to find the solution to

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ v|_{t=0} = g(x) & x \in \mathbb{R}, \\ v_t|_{t=0} = h(x) & x \in \mathbb{R}. \end{cases} \tag{8}$$

From (3), the general solution to the PDE is

$$v(x,t) = \phi(x - ct) + \psi(x + ct).$$

We now use the initial conditions to solve for ϕ and ψ. The first initial condition $v|_{t=0} = g(x)$ implies

$$v(x,t)|_{t=0} = \big(\phi(x - ct) + \psi(x + ct)\big)\big|_{t=0} = \phi(x) + \psi(x) = g(x) \quad \text{for } x \in \mathbb{R}, \tag{9}$$

and the second initial condition $v_t|_{t=0} = h(x)$ implies

$$v_t(x,t)|_{t=0} = \big(-c\phi'(x - ct) + c\psi'(x + ct)\big)\big|_{t=0} = -c\phi'(x) + c\psi'(x) = h(x) \quad \text{for } x \in \mathbb{R}. \tag{10}$$

We now differentiate (9) and compare with (10) to get the system of equations

$$\begin{cases} \phi'(x) + \psi'(x) = g'(x) \\ -c\phi'(x) + c\psi'(x) = h(x) \end{cases}.$$

Solving for ϕ implies

$$2c\phi'(x) = cg'(x) - h(x) \implies \phi(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(s)\,ds + C \quad \text{for } x \in \mathbb{R},$$

and solving for ψ implies

$$2c\psi'(x) = cg'(x) + h(x) \implies \psi(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(s)\,ds + D \quad \text{for } x \in \mathbb{R}.$$

Therefore, using (3) and the formulas for ϕ and ψ above,

$$v(x,t) = \phi(x - ct) + \psi(x + ct) = \frac{g(x + ct) + g(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds + C + D.$$

We now have to find the constants $C + D$. The equation must hold for all x and t, so plugging in $x = 0$ and $t = 0$ and using the fact $v(0,0) = g(0)$ by the first initial condition implies

$$v(0,0) = g(0) + C + D = g(0) \implies C + D = 0.$$

Therefore the particular solution to (8) (the first two terms in (4)) is

$$v(x,t) = \frac{g(x + ct) + g(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$$

Remark 2. It turns out that $C = \phi(0) - \frac{1}{2}g(0)$ and $D = \phi(0) - \frac{1}{2}g(0)$ by the fundamental theorem of calculus and our particular choice of the lower limit of integration.

1.2.2 The Particular Solution to the Inhomogeneous IVP

We will use a change of variables to find the solution to

$$\begin{cases} w_{tt} - c^2 w_{xx} = f(x,t) & x \in \mathbb{R},\ t > 0, \\ w|_{t=0} = 0 & x \in \mathbb{R}, \\ w_t|_{t=0} = 0 & x \in \mathbb{R}. \end{cases} \tag{11}$$

Characteristic Coordinates: We want to parametrize by the characteristic coordinates $\xi(x,t) = x + ct$ and $\eta(x,t) = x - ct$, so we use the change of variables

$$x = \frac{\xi + \eta}{2} \quad \text{and} \quad t = \frac{\xi - \eta}{2c}.$$

This is essentially the inverse change of variables in Method 2 (see Section 1.1.2). Under this change of variables, we have

$$\partial_\xi = \frac{1}{2}\partial_x + \frac{1}{2c}\partial_t \quad \text{and} \quad \partial_\eta = \frac{1}{2}\partial x - \frac{1}{2c}\partial_t,$$

so

$$\partial_\eta \partial_\xi = \left(\frac{1}{2}\partial x - \frac{1}{2c}\partial_t\right)\left(\frac{1}{2}\partial_x + \frac{1}{2c}\partial_t\right) = -\frac{1}{4c^2}(\partial_t^2 - c^2\partial_x^2).$$

Therefore,

$$w_{tt} - c^2 w_{xx} = f(x,t) \implies w_{\xi\eta} = -\frac{1}{4c^2} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right).$$

Integrating with respect to η then ξ, we get

$$w(\xi,\eta) = -\frac{1}{4c^2}\int_{\xi_0}^{\xi}\int_{\eta_0}^{\eta} f\left(\frac{\tilde{\xi}+\tilde{\eta}}{2}, \frac{\tilde{\xi}-\tilde{\eta}}{2c}\right) d\tilde{\eta}\, d\tilde{\xi} + \phi(\xi) + \psi(\eta),$$

where ϕ and ψ are differentiable functions. We can choose the lower limit to be anything we wish (as long as it is independent of the upper limit of integration), but a clever choice of the lower limits will make the integral part of the solution satisfy the boundary conditions $w|_{t=0} = 0$ and $w_t|_{t=0} = 0$ automatically. To this end, we take $\eta_0 = \tilde{\xi}$ and $\xi_0 = \eta$, so

$$w(\xi,\eta) = -\frac{1}{4c^2}\int_{\eta}^{\xi}\int_{\tilde{\xi}}^{\eta} f\left(\frac{\tilde{\xi}+\tilde{\eta}}{2}, \frac{\tilde{\xi}-\tilde{\eta}}{2c}\right) d\tilde{\eta}\, d\tilde{\xi} + \phi(\xi) + \psi(\eta). \tag{12}$$

This choice of lower bound is special because we can solve for $\phi(\xi)$ and $\psi(\eta)$. We now evaluate the formula when $t = 0$. In this case, we have $\xi|_{t=0} = \eta|_{t=0} = x$, so the initial condition $w|_{t=0} = 0$ implies

$$0 = w(\xi,\eta)|_{t=0} = -\frac{1}{4c^2}\int_{x}^{x}\int_{\tilde{\xi}}^{x} f\left(\frac{\tilde{\xi}+\tilde{\eta}}{2}, \frac{\tilde{\xi}-\tilde{\eta}}{2c}\right) d\tilde{\eta}\, d\tilde{\xi} + \big(\phi(\xi) + \psi(\eta)\big)|_{t=0}.$$

The integral term vanishes and $(\phi(\xi) + \psi(\eta))|_{t=0} = (\phi(x) + \psi(x))$, so we can conclude

$$\phi(x) + \psi(x) = 0 \quad \text{for all } x \in \mathbb{R}. \tag{13}$$

Next, we use the Leibniz integral rule to differentiate with respect to t and use the fact that $w_t|_{t=0} = 0$ to conclude that

$$\begin{aligned} 0 = w_t(\xi,\eta)|_{t=0} = \Bigg(&-\frac{1}{4c}\int_{\xi}^{\eta} f\left(\frac{\xi+\tilde{\eta}}{2}, \frac{\xi-\tilde{\eta}}{2c}\right) d\tilde{\eta} - \frac{1}{4c}\int_{\eta}^{\eta} f\left(\frac{\eta+\tilde{\eta}}{2}, \frac{\eta-\tilde{\eta}}{2c}\right) d\tilde{\eta} \\ &+ \frac{1}{4c}\int_{\eta}^{\xi} f\left(\frac{\tilde{\xi}+\eta}{2}, \frac{\tilde{\xi}-\eta}{2c}\right) d\tilde{\xi} + \big(c\phi'(\xi) - c\psi'(\eta)\big)\Bigg)\Bigg|_{t=0}. \end{aligned}$$

Since $\xi|_{t=0} = \eta|_{t=0} = x$, all the integral terms vanish leaving us with

$$c\phi'(x) - c\psi'(x) = 0 \quad \text{for all } x \in \mathbb{R}. \tag{14}$$

Integrating (14) and comparing it with (13) implies that

$$\begin{cases} \phi(x) + \psi(x) = 0 \\ \phi(x) - \psi(x) = C \end{cases} \implies \phi(x) = \frac{C}{2},\ \psi(x) = -\frac{C}{2} \quad \text{for all } x \in \mathbb{R}.$$

Therefore, $\phi(\xi) + \psi(\eta) = -\frac{C}{2} + \frac{C}{2} = 0$, so the particular solution (12) to (11) expressed in terms of the characteristic coordinates simplifies to

$$w(\xi,\eta) = \frac{1}{4c^2}\int_\eta^\xi \int_\eta^{\tilde{\xi}} f\left(\frac{\tilde{\xi}+\tilde{\eta}}{2}, \frac{\tilde{\xi}-\tilde{\eta}}{2c}\right) d\tilde{\eta}\, d\tilde{\xi}. \tag{15}$$

Original Coordinates: We now have to write (15) back in terms of x and t. Notice that the region of integration in (15) corresponds to the triangle

$$\tilde{\Delta}(\xi,\eta) = \{(\tilde{\xi},\tilde{\eta}) : \eta \le \tilde{\eta} \le \tilde{\xi} \le \xi\}.$$

Using the change of variables $\tilde{\xi} = y + cs$ and $\tilde{\eta} = y - cs$, this triangle in the original coordinates is

$$\Delta(x,t) = \{(y,s) : x - ct \le y - cs \le y + cs \le x + ct\}.$$

Since

$$x - ct \le y - cs \implies x - c(t-s) \le y \quad \text{and} \quad y + cs \le x + ct \implies y \le x + c(t-s)$$

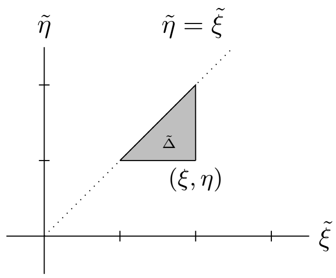
and for $y = x$

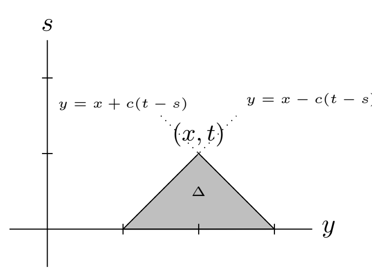
$$x - ct \le x - cs \le x + cs \le x + ct \implies -t \le -s \le s \le t \implies 0 \le s \le t,$$

the region of integration (also called the *domain of dependence*) can be simplified to

$$\Delta(x,t) = \{(y,s) : 0 \le s \le t,\ x - c(t-s) \le y \le x + c(t-s)\}.$$

The regions of integrations $\tilde{\Delta}$ and Δ are plotted below:





The Jacobian of the transformation is

$$d\tilde{\eta}\, d\tilde{\xi} = \left|\det\begin{pmatrix} 1 & -c \\ 1 & c \end{pmatrix}\right| dyds = 2c\ dyds, \tag{16}$$

so the solution to (11) using (15) can be expressed as

$$w(x,t) = \frac{1}{2c}\iint_\Delta f(y,s)\ dyds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dyds.$$

giving us the third term in (4).

Remark 3. The assumptions on the regularity of g, h and f ensure that $u \in C^2(\mathbb{R})$. If we weaken the notion of what it means to be a solution, then we can assume less conditions on g, h, and f. We can check that the formula extends to these cases when the initial conditions might not be differentiable.

1.3 Example Problems

Problem 1.1. ($\star$) Solve the initial value problem

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = \tanh(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = \arctan(x) & x \in \mathbb{R}. \end{cases}$$

Solution 1.1. By d'Alembert's formula (4), the particular solution to this IVP is given by

$$u(x,t) = \frac{\tanh(x+2t) + \tanh(x-2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \arctan(y)\,dy.$$

The integral term can be computed using integration by parts,

$$\begin{aligned} &\frac{1}{4}\int_{x-2t}^{x+2t} \arctan(y)\,dy \\ &= \frac{1}{4}\Big(y\arctan(y) - \frac{1}{2}\ln|1+y^2|\Big)\Big|_{y=x-2t}^{y=x+2t} \\ &= \frac{1}{4}\Big((x+2t)\arctan(x+2t) - (x-2t)\arctan(x-2t) - \frac{1}{2}\ln(1+(x+2t)^2) + \frac{1}{2}\ln(1+(x-2t)^2)\Big). \end{aligned}$$

Problem 1.2. ($\star$) Solve the following initial value problems

1. $$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R} \end{cases}$$

   with

   $$g(x) = \begin{cases} 0 & |x| \geq 1 \\ x^2 - x^4 & |x| < 1 \end{cases}, \qquad h(x) = 0.$$

2. $$\begin{cases} u_{tt} - 4u_{xx} = 0 & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R} \end{cases}$$

   with

   $$g(x) = 0, \qquad h(x) = \begin{cases} 0 & |x| \geq 1 \\ x^2 - x^4 & |x| < 1 \end{cases}.$$

Solution 1.2.

(1) Since $h(x) = 0$, by d'Alembert's formula (4), the particular solution to this IVP is given by

$$u(x,t) = \frac{g(x+2t) + g(x-2t)}{2}.$$

Since $g(x)$ changes form based on the value of $|x|$, we can break our solution into 4 cases:

A. $|x+2t| \geq 1$, $|x-2t| \geq 1$: On this region, $g(x+2t) = 0$ and $g(x-2t) = 0$, so

$$u(x,t) = 0.$$

B. $|x+2t| < 1$, $|x-2t| \geq 1$: On this region, $g(x+2t) = (x+2t)^2 - (x+2t)^4$ and $g(x-2t) = 0$, so

$$u(x,t) = \frac{(x+2t)^2 - (x+2t)^4}{2}.$$

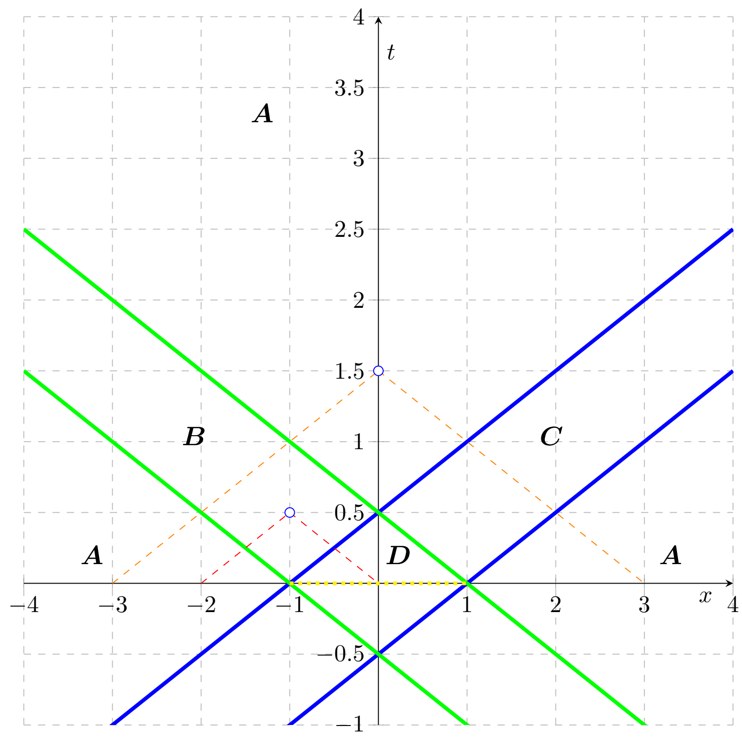
C. $|x+2t| \geq 1$, $|x-2t| < 1$: On this region, $g(x+2t) = 0$ and $g(x-2t) = (x-2t)^2 - (x-2t)^4$, so

$$u(x,t) = \frac{(x-2t)^2 - (x-2t)^4}{2}.$$

D. $|x+2t| < 1$, $|x-2t| < 1$: On this region, $g(x+2t) = (x+2t)^2 - (x+2t)^4$ and $g(x-2t) = (x-2t)^2 - (x-2t)^4$, so

$$u(x,t) = \frac{(x+2t)^2 - (x+2t)^4 + (x-2t)^2 - (x-2t)^4}{2}.$$

Characteristic Lines: The regions A, B, C, D are displayed below



Description of Picture: The initial condition is supported on the interval $[-1,1]$. The wave propagates right along the lines $x - 2t = C \in [-1,1]$ (between the blue characteristic lines) and left along the lines $x + 2t = C \in [-1,1]$ (between the green characteristic lines). The behavior on each of the regions can be determined by drawing the domain of dependence at the point (x,t) and seeing if the corners lie in the interval $[-1,1]$. For example, at the point $(-1,0.5)$ the left corner does not lie in $[-1,1]$, while the right corner is in $[-1,1]$, which corresponds to case B above. Similarly, at the point $(0,1.5)$ both corners do not lie in $[-1,1]$, which corresponds to case A above.

(2) Since $g(x) = 0$, by d'Alembert's formula (4), the particular solution to this IVP is given by

$$u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy.$$

Since $h(x)$ changes form based on the value of $|x|$, we can break our solution into 5 cases:

A. $x - 2t \le -1 \le 1 \le x + 2t$: On this region, we can split our region of integration into

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{-1} h(y)\,dy + \frac{1}{4}\int_{-1}^{1} h(y)\,dy + \frac{1}{4}\int_{1}^{x+2t} h(y)\,dy \\ &= \frac{1}{4}\int_{-1}^{1} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=-1}^{y=1} = \frac{1}{15}. \end{aligned}$$

B. $x - 2t \le -1 \le x + 2t \le 1$: On this region, we can split our region of integration into

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{-1} h(y)\,dy + \frac{1}{4}\int_{-1}^{x+2t} h(y)\,dy \\ &= \frac{1}{4}\int_{-1}^{x+2t} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=-1}^{y=x+2t} = \frac{(x+2t)^3}{12} - \frac{(x+2t)^5}{20} + \frac{1}{30}. \end{aligned}$$

C. $-1 \le x - 2t \le 1 \le x + 2t$: On this region, we can split our region of integration into

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{1} h(y)\,dy + \frac{1}{4}\int_{1}^{x+2t} h(y)\,dy \\ &= \frac{1}{4}\int_{x-2t}^{1} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=x-2t}^{y=1} = \frac{1}{30} - \frac{(x-2t)^3}{12} + \frac{(x-2t)^5}{20}. \end{aligned}$$

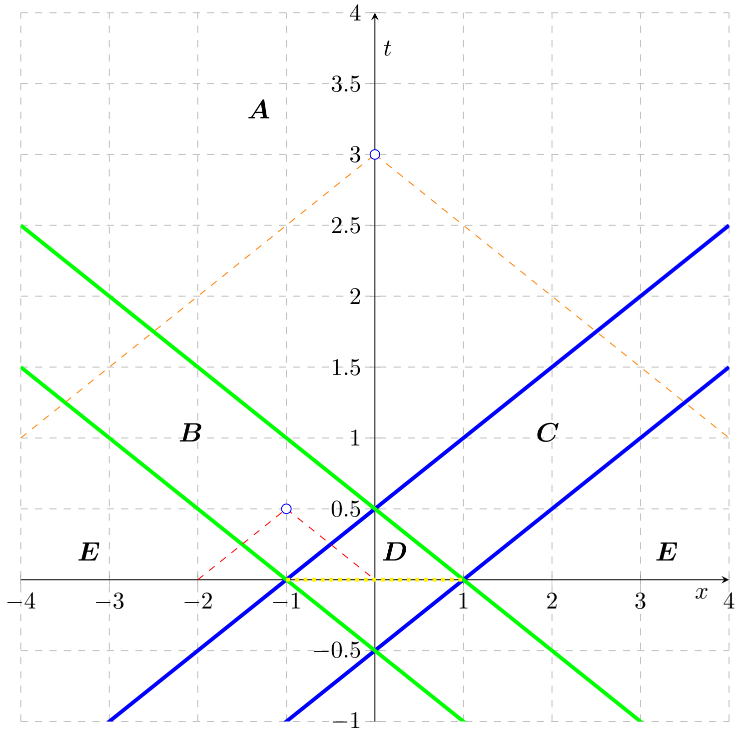
D. $-1 \le x - 2t \le x + 2t \le 1$: On this region, the integrand is always equal to $h(y) = y^2 - y^4$

$$\begin{aligned} u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} h(y)\,dy &= \frac{1}{4}\int_{x-2t}^{x+2t} y^2 - y^4\,dy \\ &= \frac{1}{4}\Big(\frac{y^3}{3} - \frac{y^5}{5}\Big)\Big|_{y=x-2t}^{y=x+2t} \\ &= \frac{(x+2t)^3}{12} - \frac{(x+2t)^5}{20} - \frac{(x-2t)^3}{12} + \frac{(x-2t)^5}{20}. \end{aligned}$$

E. $x - 2t \ge 1$, or $x + 2t \le -1$: On this region, the integrand is always equal to $h(y) = 0$, so

$$u(x,t) = 0.$$

Characteristic Lines: The regions A, B, C, D, E are displayed below



Description of Picture: The initial condition is supported on the interval $[-1,1]$. The behavior in each of the regions can be determined by drawing the domain of dependence at the point (x,t) and seeing how much of the interval $[-1,1]$ is contained in the base of the triangle. For example, at $(-1,0.5)$ the left corner of the base of the triangle is < -1 and the right corner of the base is in $[-1,1]$, which corresponds to case B above. Similarly, at $(0,3)$ the left corner of the base of the orange triangle is < -1 and the right corner of the base is in > 1, which corresponds to case A above.

Problem 1.3. $(\star)$ Solve the initial value problem

$$\begin{cases} u_{tt} - 4u_{xx} = f(x,t) & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}, \\ u_t|_{t=0} = h(x) & x \in \mathbb{R} \end{cases}$$

with

$$f(x,t) = \begin{cases} \sin(x) & 0 < t < \pi \\ 0 & t \geq \pi \end{cases}, \qquad g(x) = 0, \qquad h(x) = 0.$$

Solution 1.3. Since $g(x) = 0$ and $h(x) = 0$, by d'Alembert's formula (4) the particular solution to this IVP is given by

$$u(x,t) = \frac{1}{4} \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y) \mathbb{1}_{[0,\pi]}(s)\, dyds = \frac{1}{4} \int_0^{\min(t,\pi)} \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\, dyds.$$

If you draw the region of integration, we are basically chopping off Δ above the line $t = \pi$ and integrating the remaining trapezoid (or triangle if t is small enough). We have two cases,

A. $t < \pi$: On this region, we have

$$\begin{aligned} u(x,t) &= \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\,dyds \\ &= \frac{1}{4}\int_0^t \left(-\cos(y)\Big|_{y=x-2(t-s)}^{y=x+2(t-s)} \right) ds \\ &= \frac{1}{4}\int_0^t -\cos(x+2(t-s)) + \cos(x-2(t-s))ds. \\ &= \frac{1}{8}\Big(\sin(x+2(t-s)) + \sin(x-2(t-s)) \Big)\Big|_{s=0}^{s=t} \\ &= \frac{1}{4}\sin(x) - \frac{1}{8}\sin(x+2t) - \frac{1}{8}\sin(x-2t). \end{aligned}$$

B. $t \geq \pi$: On this region, we have

$$\begin{aligned} u(x,t) &= \frac{1}{4}\int_0^\pi \int_{x-2(t-s)}^{x+2(t-s)} \sin(y)\,dyds \\ &= \frac{1}{4}\int_0^\pi \left(-\cos(y)\Big|_{y=x-2(t-s)}^{y=x+2(t-s)} \right) ds \\ &= \frac{1}{4}\int_0^\pi -\cos(x+2(t-s)) + \cos(x-2(t-s))ds. \\ &= \frac{1}{8}\Big(\sin(x+2(t-s)) + \sin(x-2(t-s)) \Big)\Big|_{s=0}^{s=\pi} \\ &= 0. \end{aligned}$$

Problem 1.4. ($\star\star$) Find the solution to the *Goursat* problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > c|t|; \\ u|_{x=-ct} = g(t), & t < 0; \\ u|_{x=ct} = h(t), & t > 0, \end{cases}$$

where g and h satisfy the compatibility condition $g(0) = h(0)$.

Solution 1.4. Recall that the general solution of $u_{tt} - c^2 u_{xx} = 0$ is given by

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \text{ for } x > c|t|,$$

for some yet to be determined functions ϕ and ψ. Using the initial conditions, we can recover the specific form of ϕ and ψ. For $t < 0$, the first boundary condition implies,

$$u|_{x=-ct} = g(t) \implies \phi(0) + \psi(-2ct) = g(t) \overset{s=-2ct}{\implies} \psi(s) = g\Big(-\frac{s}{2c}\Big) - \phi(0) \text{ for } s > 0$$

and for $t > 0$, the second boundary condition implies

$$u|_{x=ct} = h(t) \implies \phi(2ct) + \psi(0) = h(t) \overset{s=2ct}{\implies} \phi(s) = h\Big(\frac{s}{2c}\Big) - \psi(0) \text{ for } s > 0.$$

If we take limits as $s \to 0$ from right, the condition $g(0) = h(0)$ implies that

$$\psi(0) = g(0) - \phi(0) \quad \text{and} \quad \phi(0) = h(0) - \psi(0) \implies \psi(0) + \phi(0) = g(0) = h(0) = \frac{g(0)+h(0)}{2}.$$

Therefore, our particular solution is given by

$$u(x,t) = h\Big(\frac{x+ct}{2c}\Big) + g\Big(\frac{ct-x}{2c}\Big) - (\phi(0)+\psi(0)) = h\Big(\frac{x+ct}{2c}\Big) + g\Big(\frac{ct-x}{2c}\Big) - \frac{g(0)+h(0)}{2}, \tag{17}$$

since $x + ct > 0$ and $ct - x < 0$ for $x > c|t|$, the solution is uniquely defined on this region.

Problem 1.5. ($\star\star\star$) Find the solution to the *Goursat* problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > c|t|; \\ u|_{x=-ct} = g(t), & t < 0; \\ u|_{x=ct} = h(t), & t > 0, \end{cases}$$

where g and h satisfy the compatibility condition $g(0) = h(0)$.

Solution 1.5. This is the inhomogeneous variant of Problem 1.4.

Inhomogeneous solution: We computed the homogeneous solution in the previous exercise. It suffices to find the solution to the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > c|t|; \\ u|_{x=-ct} = 0, & t < 0; \\ u|_{x=ct} = 0, & t > 0. \end{cases}$$

Since we want to parametrize by the characteristic coordinates $\xi = x + ct$ and $\eta = x - ct$, we use the change of variables

$$x = \frac{\xi+\eta}{2} \quad \text{and} \quad t = \frac{\xi-\eta}{2c}.$$

Under this change of variables,

$$\partial_\xi = \frac{1}{2}\partial_x + \frac{1}{2c}\partial_t \quad \text{and} \quad \partial_\eta = \frac{1}{2}\partial x - \frac{1}{2c}\partial_t,$$

so

$$\partial_\eta\partial_\xi = \Big(\frac{1}{2}\partial x - \frac{1}{2c}\partial_t\Big)\Big(\frac{1}{2}\partial_x + \frac{1}{2c}\partial_t\Big) = -\frac{1}{4c^2}(\partial_t^2 - c^2\partial_x^2).$$

Therefore,

$$u_{tt} - c^2 u_{xx} = f(x,t) \implies u_{\xi\eta} = -\frac{1}{4c^2} f\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\Big).$$

Integrating with respect to η then ξ, we get

$$u(\xi,\eta) = -\frac{1}{4c^2}\int_{\xi_0}^{\xi}\int_{\eta_0}^{\eta} f\Big(\frac{\tilde\xi+\tilde\eta}{2}, \frac{\tilde\xi-\tilde\eta}{2c}\Big)\, d\tilde\eta\, d\tilde\xi + \phi(\xi) + \psi(\eta),$$

where ϕ and ψ are differentiable functions. We can choose the lower limit to be anything we wish, so we choose $\xi_0 = 0$ and $\eta_0 = 0$ (this particular choice will become apparent later on in the computation),

$$u(\xi,\eta) = -\frac{1}{4c^2}\int_{0}^{\xi}\int_{0}^{\eta} f\Big(\frac{\tilde\xi+\tilde\eta}{2}, \frac{\tilde\xi-\tilde\eta}{2c}\Big)\, d\tilde\eta\, d\tilde\xi + \phi(\xi) + \psi(\eta).$$

We now use the initial conditions to solve for ϕ and ψ. When $\xi = 0$, we must have $x = -ct$. On this line the initial condition $u|_{x=-ct} = 0$ implies that $u(0,\eta)$ must be 0 for all η, so

$$0 = u(0,\eta) = \phi(0) + \psi(\eta) \implies \psi(\eta) = -\phi(0).$$

Similarly, when $\eta = 0$, we must have $x = ct$. On this line the initial condition $u|_{x=ct} = 0$, so

$$0 = u(\xi, 0) = \phi(\xi) + \psi(0) \implies \phi(\xi) = -\psi(0).$$

Therefore, both $\phi(\xi)$ and $\psi(\eta)$ are constant functions, so adding these two conditions implies that

$$\phi(\xi) + \psi(\eta) = -\phi(0) - \psi(0) = -(\phi(\xi) + \psi(\eta)) \implies \phi(\xi) + \psi(\eta) = 0.$$

Since the $\phi(\xi) + \psi(\eta)$ term vanishes, changing back into the x and t coordinates (the Jacobian of this linear transformation is $2c$ by (16)), we see that

$$u(\xi, \eta) = -\frac{1}{4c^2} \int_0^{\xi} \int_0^{\eta} f\Big(\frac{\tilde{\xi} + \tilde{\eta}}{2}, \frac{\tilde{\xi} - \tilde{\eta}}{2c}\Big) d\tilde{\eta} d\tilde{\xi} \iff u(x,t) = -\frac{1}{2c} \iint_{R(x,t)} f(y,s) dy ds \tag{18}$$

where $R(x,t)$ is the image of the rectangle in $\tilde{\xi}$ and $\tilde{\eta}$,

$$\{(\tilde{\xi}, \tilde{\eta}) : 0 \le \tilde{\xi} \le \xi, 0 \le \tilde{\eta} \le \eta\} \mapsto R(x,t) = \{(y,s) : 0 \le y + cs \le x + ct, 0 \le y - cs \le x - ct\}.$$

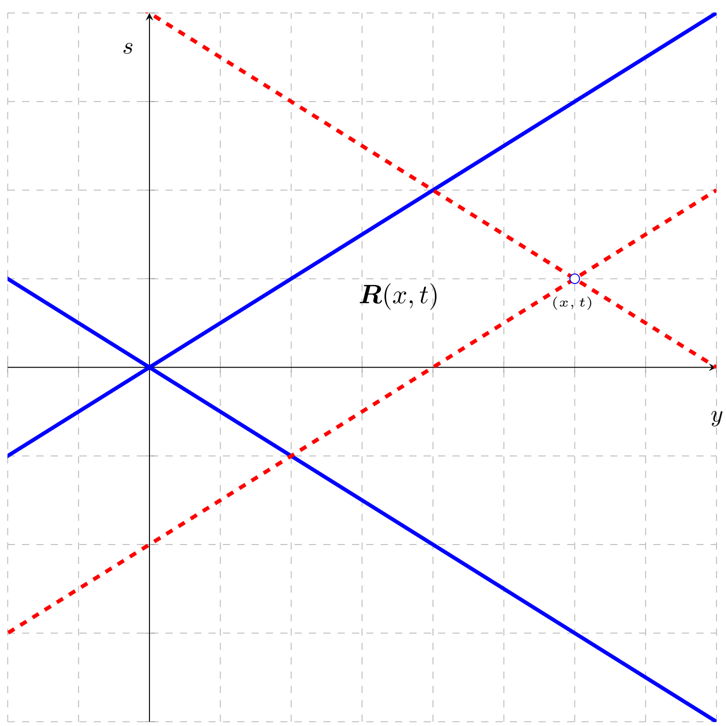
Full Solution: By linearity, the full solution of the inhomogeneous Goursat problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & x > c|t|; \\ u|_{x=-ct} = g(t), & t < 0; \\ u|_{x=ct} = h(t), & t > 0, \end{cases}$$

is given by the sum of the homogeneous (17) and inhomogeneous (18) solutions of the Goursat problem,

$$u(x,t) = h\Big(\frac{x+ct}{2c}\Big) + g\Big(\frac{ct-x}{2c}\Big) - \frac{g(0)+h(0)}{2} - \frac{1}{2c} \iint_{R(x,t)} f(y,s)\, dy ds.$$

Characteristic Lines:



Description of Picture: The region of integration $R(x,t)$ is given by the region bounded by the boundary $x = ct$ and $x = -ct$ and the characteristic lines passing through the point (x,t). In the picture above, the region of integration corresponding to the point (x,t) indicated by the blue hollow dot is the region bounded by the blue and dashed red lines.