# 1 Introduction to Partial Differential Equations

Recall that an ordinary differential equation (ODE) of order $k$ is an equation involving a single variable function $u(x)$, its derivatives, and the variable $x$,

$$F(u^{(k)}, u^{(k-1)}, \dots, u, x) = 0.$$

**Definition 1.** A partial differential equation (PDE) of order $k$ is an equation involving a multivariable function $u : \mathbb{R}^n \to \mathbb{R}$, its partial derivatives, and the independent variable $\vec{x} \in \mathbb{R}^n$ (one of these variables might be a time variable $t$),

$$F(D^k u, D^{k-1} u, \dots, u, \vec{x}) = F(\underbrace{u_{x_1 \dots x_1}}_{kth\ deriv}, \dots, u_{x_1x_1}, u_{x_1x_2}, \dots u_{x_nx_n}, u_{x_1}, \dots, u_{x_n}, \dots, u, \vec{x}) = 0. \quad (1)$$

The *order* of the PDE is the order of the highest derivative in the equation. The function $u$ is called a *solution* if $u$ satisfies (1) in some region in $\mathbb{R}^n$.

**Example 1.** The following second order PDE

$$u_{xy} + x = 0$$

has general solution

$$u = -\frac{yx^2}{2} + f(x) + g(y)$$

where $f$ and $g$ are arbitrary differentiable functions.

## 1.1 Boundary and Initial Value Problems

The general solutions of PDEs are usually not unique. We impose some additional conditions such as a domain, boundary values, or initial values to give solutions some additional nice properties.

**Definition 2.** A PDE is *well-posed* if it satisfies:

1. Existence: There is at least one solution $u(\vec{x})$ satisfying the PDE and the additional conditions.
2. Unique: There is at most one solution $u(\vec{x})$ satisfying the PDE and the additional conditions.
3. Stability: The solution depends continuously on the initial data. This means that if the conditions are changed a little, the corresponding solution changes only a little.

**Example 2.** The following second order PDE

$$u_{xy} + x = 0 \qquad \text{for } x, y > 0,$$

with boundary conditions

$$u|_{y=0} = 0, \qquad u|_{x=0} = 0$$

has the particular solution

$$u = -\frac{yx^2}{2}.$$

If we give too few constraints, then we might not get a unique solution. If we give too many constraints, then a solution may fail to exist.

**Definition 3.** We have the following terminology for these different types of constraints:

1. Initial Value Problem (IVP): A constraint on the time variable is put at $t = 0$, e.g. $u|_{t=0} = f(\vec{x})$.
2. Boundary Value Problem (BVP): A constraint on the spacial variable is put on the boundary of the domain $\Omega$, e.g. $u|_{\partial\Omega} = f(\vec{x})$.
3. Initial Boundary Value Problem (IBVP): A constraint on both the time and space variables.

## 1.2 Classification

**Definition 4.** An operator $L$ is *linear* if for every pair of functions $u, v$ and numbers $s, t$

$$L[su + tv] = sL[u] + tL[v].$$

**Example 3.** The operator $L = x^2\partial_x + \partial_{yy}$ is linear because

$$L[su + tv] = x^2\partial_x(su + tv) + \partial_{yy}(su + tv) = sx^2u_x + tx^2v_x + su_{yy} + tv_{yy} = sL[u] + tL[v].$$

A PDE $L[u] = f(\vec{x})$ is linear if $L$ is a linear operator. Nonlinear PDE can be classified based on how close it is to being linear. Let $F$ be a nonlinear function and $\alpha = (\alpha_1, \dots, \alpha_n)$ denote a multi-index.:

1. Linear: A PDE is *linear* if the coefficients in front of the partial derivative terms are all functions of the independent variable $\vec{x} \in \mathbb{R}^n$,

$$\sum_{|\alpha|\leq k} a_\alpha(\vec{x})D^\alpha u = f(\vec{x}).$$

   A linear PDE is *homogeneous* if there is no term that depends only on the space variables, i.e. $f = 0$. Likewise, a linear PDE is *inhomogeneous* if $f \neq 0$.

2. Semilinear: A PDE is *semilinear* if it is nonlinear, but the coefficients in front of the highest order partial derivative terms are all functions of the independent variable $\vec{x} \in \mathbb{R}^n$,

$$\sum_{|\alpha|=k} a_\alpha(\vec{x})D^\alpha u + F(D^{k-1}u, \dots, Du, u, \vec{x}) = 0.$$

3. Quasilinear: A PDE is *quasilinear* if it is nonlinear, but the coefficients in front of the highest order partial derivative terms are all functions of the independent variable $\vec{x} \in \mathbb{R}^n$ or lower derivative terms,

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, Du, u, \vec{x})D^\alpha u + F(D^{k-1}u, \dots, Du, u, \vec{x}) = 0.$$

4. Fully Nonlinear: A PDE is *fully nonlinear* if it is not of the above 3 forms. That is, the PDE is fully nonlinear if it depends nonlinearly on the highest order partial derivative terms,

$$F(D^k u, \dots, Du, u, \vec{x}) = 0.$$

**Example 4.** The forms of the various types first order PDE in $\mathbb{R}^2$ are:

1. Linear Homogeneous:

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0$$

2. Linear Inhomogeneous:

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

3. Semilinear:

$$a(x, y)u_x + b(x, y)u_y + F(u, x, y) = 0$$

4. Quasilinear:

$$a(x, y, u)u_x + b(x, y, u)u_y + F(u, x, y) = 0$$

5. Fully Nonlinear:

$$F(x, y, u, u_x, u_y) = 0$$

## 1.3 Classification of Linear Second Order Partial Differential Equations

Consider a general linear homogeneous second order PDE

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0. \tag{2}$$

where the coefficients $a_{11}, \dots, a_0$ are functions of $x, y$. We can do a linear transformation of the $x, y$ variables to simplify PDE by removing the $u_{xy}$ term. There are 3 types of second order PDEs:

1. Elliptic: If $a_{12}^2 - 4a_{11}a_{22} < 0$, then (2) can be reduced to the form

$$u_{xx} + u_{yy} + \text{lower order terms} = 0.$$

2. Hyperbolic: If $a_{12}^2 - 4a_{11}a_{22} > 0$, then (2) can be reduced to the form

$$u_{xx} - u_{yy} + \text{lower order terms} = 0.$$

3. Parabolic: If $a_{12}^2 - 4a_{11}a_{22} = 0$, then (2) can be reduced to the form

$$u_{xx} + \text{lower order terms} = 0.$$

**Example 5.** The linear second order PDE

$$4u_{xx} - 6u_{xy} + 9u_{yy} + u_x + u_y + u = 0$$

is elliptic because $(-6)^2 - 4 \cdot 4 \cdot 9 < 0$.

**Remark 1.** The naming of these second order equations coincide with the conic sections. Consider the quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + C = 0.$$

The type of equation can be analyzed by looking at the discriminant $\Delta = B^2 - 4AC$:

(1) Ellipse: $B^2 - 4AC < 0$ (2) Hyperbola: $B^2 - 4AC > 0$ (3) Parabola: $B^2 - 4AC = 0$.

## 1.4 Examples of PDEs

1. **Transport Equation:**

$$u_t + cu_x = 0$$

2. **Wave Equation:**

$$u_{tt} - c^2u_{xx} = 0$$

3. **Heat Equation:**

$$u_t - ku_{xx} = 0$$

4. **Laplace's Equation:**

$$u_{xx} + u_{yy} = 0$$

5. **Minimal Surface Equation:**

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0$$

## 1.5 Example Problems

### 1.5.1 Classification

**Problem 1.1.** ($\star$) Consider first order equations and determine if they are linear homogeneous, linear inhomogeneous, or nonlinear; for nonlinear equations, indicate if they are also semilinear, or quasilinear:

$$u_y + xu_x - u = 0, \tag{A}$$
$$u_y + u_x - u^2 = 0, \tag{B}$$
$$u_y + uu_x + x = 0, \tag{C}$$
$$2\sin(u_y) + u_x = 0. \tag{D}$$

**Solution 1.1.**

(A) Since the coefficients in front of $u_y, u_x$, and $u$ are functions of $x$ and $y$ only, the equation is linear. There is also no term that depends only on $x$ or $y$, so it is homogeneous. To prove that the equation is linear, notice that

$$\begin{aligned} L[au + bv] &= (au + bv)_y + x(au + bv)_x - (au + bv) \\ &= a(u_y + xu_x - u) + b(v_y + xv_x - v) \\ &= aL[u] + bL[v]. \end{aligned}$$

(B) There is a $u^2$ term, so the function is nonlinear. However, the coefficients of the highest order terms are functions of $x$ and $y$, so the equation is semilinear. To prove that the operator is nonlinear, we show that the scaling property fails for a nice function such as $u(x, y) = x$,

$$L[2x] = (2x)_y + (2x)_x - (2x)^2 = 2 - 4x^2 \neq 2 - 2x^2 = 2((x)_y + (x)_x - x^2) = 2L[x].$$

(C) There is a $uu_x$ term, so the function is nonlinear. However, the coefficients of the highest order terms are functions of $x$, $y$ and $u$, so the equation is quasilinear.

(D) The nonlinear term $\sin(u_y)$ is with respect to the highest order term, so it is fully nonlinear.

### 1.5.2 Multivariable Antidifferentiation

**Problem 1.2.** ($\star$) Find the general solutions to the following equations:

$$u_{xxy} = 0, \tag{1}$$
$$u_{xyz} = \sin(x) + \sin(y)\sin(z). \tag{2}$$

**Solution 1.2.**

(1) We integrate out each of the partial derivatives by treating the remaining variables as constants and introduce an integration constant in each step,

$$\begin{aligned} & u_{xxy} = 0 \\ \Rightarrow\ & u_{xx} = f_{xx}(x) \\ \Rightarrow\ & u_x = f_x(x) + g(y) \\ \Rightarrow\ & u = f(x) + xg(y) + h(y) \end{aligned}$$

where $f(x)$ is a twice differentiable function.

(2) We integrate out each of the partial derivatives by treating the remaining variables as constants and introduce an integration constant in each step,

$$\begin{aligned} u_{xyz} &= \sin(x) + \sin(y)\sin(z) \\ \Rightarrow\ u_{xy} &= z\sin(x) - \sin(y)\cos(z) + f_{xy}(x,y) \\ \Rightarrow\ u_x &= yz\sin(x) + \cos(y)\cos(z) + f_x(x,y) + g_x(x,z) \\ \Rightarrow\ u &= -yz\cos(x) + x\cos(y)\cos(z) + f(x,y) + g(x,z) + h(y,z) \end{aligned}$$

where $f(x,y)$ is differentiable in $x$ and $y$, and $g(x,z)$ is differentiable in $x$.

**Problem 1.3.** ($\star\star$) Find the general solution to

$$u_{xy} = 2u_x + e^{x+y}.$$

**Solution 1.3.** To simplify notation, we define $v(x,y) = u_x(x,y)$. Treating $x$ as a constant, we first solve the ODE

$$v_y = 2v + e^{x+y} \implies v_y - 2v = e^{x+y}.$$

This is a linear inhomogeneous ODE in $y$, so it can be solved using the integrating factor

$$I(y) = e^{\int -2\,dy} = e^{-2y}.$$

We multiply both sides by $e^{-2y}$ and integrate to solve for $v$,

$$e^{-2y}v_y - 2e^{-2y}v = e^{x-y} \Rightarrow (e^{-2y}v)_y = e^{x-y} \Rightarrow e^{-2y}v = -e^{x-y} + f_x(x) \Rightarrow v = -e^{x+y} + e^{2y}f_x(x).$$

Since $v = u_x$, we can now integrate in $x$ to recover $u$,

$$u_x = -e^{x+y} + e^{2y}f_x(x) \implies u = -e^{x+y} + f(x)e^{2y} + g(y),$$

where $f(x)$ is a differentiable function.

### 1.5.3 Properties of Solutions

**Problem 1.4.** ($\star\star\star$) Show that

$$u_n(x,y) = \exp(ny - \sqrt{n})\sin nx \tag{3}$$

is a solution of the following BVP

$$(*) \quad \begin{cases} u_{xx} + u_{yy} = 0 & x \in \mathbb{R} \quad y > 0 \\ u(0,y) = 0 & u_y(x,0) = n\exp(-\sqrt{n})\sin nx \end{cases}$$

for positive integer $n$. Show that this problem is not stable.

**Solution 1.4.**

*Checking the Solution:* We just differentiate $u_n$ and see

$$\begin{aligned} \partial_{xx}u_n(x,y) &= -n^2\exp(ny - \sqrt{n})\sin nx \\ \partial_{yy}u_n(x,y) &= n^2\exp(ny - \sqrt{n})\sin nx \end{aligned}$$

and so $u_{xx} + u_{yy} = 0$. The initial conditions are also easy to check

$$u_n(0, y) = \exp(ny - \sqrt{n}) \sin nx\Big|_{x=0} = 0$$
$$\partial_y u_n(0, y) = n \exp(ny - \sqrt{n}) \sin nx\Big|_{y=0} = n \exp(-\sqrt{n}) \sin nx.$$

Therefore, $u_n$ solves $(*)$.

*Not Stable:* We will show that the solutions $u_n(x, t)$ are not stable. If the initial conditions converge, we would expect the associated solutions to converge as well. We will show that this does not happen.

Clearly, the solution to the limiting BVP

$$(\dagger) \quad \begin{cases} u_{xx} + u_{yy} = 0 & x \in \mathbb{R} \quad y > 0 \\ u(0, y) = 0 & u_y(x, 0) = 0 \end{cases}$$

is $u_\infty(x, y) \equiv 0$. Notice the initial condition function

$$\sup_{x\in\mathbb{R}} |n \exp(-\sqrt{n}) \sin nx| = \|n \exp(-\sqrt{n}) \sin nx\|_\infty \to 0.$$

Therefore, for any $\epsilon > 0$ there exists $n$ sufficiently large so that

$$\|n \exp(-\sqrt{n}) \sin nx\|_\infty \leq \epsilon.$$

Therefore, for $n$ large, we see that

$$(\dagger\dagger) \quad \begin{cases} u_{xx} + u_{yy} = 0 & x \in \mathbb{R} \quad y > 0 \\ u(0, y) = 0 & u_y(x, 0) = n \exp(-\sqrt{n}) \sin nx \end{cases}$$

is a small change in the boundary conditions in $(\dagger)$, in the sense that the maximum distance between the initial conditions are bounded by $\epsilon$. From the above computations, we know that $u_n(x, y)$ defined in (3) is a solution to $(\dagger\dagger)$.

If the problem was well-posed, we would expect the solution $u_n$ of $(\dagger\dagger)$ to be close to the solution $u_\infty$ of $(\dagger)$ as $n$ gets large. However, on any region $\mathbb{R} \times [0, \delta]$ around the origin

$$\|u_\infty(x, y) - u_n(x, y)\|_\infty = \max_{(x,y)\in\mathbb{R}\times[0,\delta]} |\exp(ny - \sqrt{n}) \sin nx| \geq |\exp(n\delta - \sqrt{n})|,$$

which can be made arbitrarily large by taking $n$ sufficiently large. That is, $u_n(x, y) \not\to u_\infty(x, y)$ uniformly on $\mathbb{R} \times [0, \delta]$.

**Remark 2.** The PDE is not stable in the sense that we can change the magnitude of the initial conditions by an arbitrarily small amount, but the solution to the perturbed PDE $(\dagger\dagger)$ is very far apart from the original PDE $(\dagger)$ even at a small distance away from $\mathbb{R} \times \{0\}$.

**Remark 3.** We essentially showed that the solution does not depend continuously on the initial data. We can think of the solution to $(\star)$ as a function of the initial conditions. We showed that there exists an $\epsilon > 0$ such that for all $\delta > 0$, we can find initial conditions $f_1$ and $f_2$ so that

$$\|f_1 - f_2\|_\infty < \delta,$$

but the corresponding solutions satisfies

$$\|u_{f_1} - u_{f_2}\|_\infty > \epsilon,$$

violating the notion of continuity on the space of functions with the uniform norm.