1 Fourier Transform

We introduce the concept of Fourier transforms. This extends the Fourier method for finite intervals to infinite domains. In this section, we will derive the Fourier transform and its basic properties.

1.1 Heuristic Derivation of Fourier Transforms

1.1.1 Complex Full Fourier Series

Recall that *DeMoivre formula* implies that

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

This implies that the set of eigenfunctions for the full Fourier series on $[-L, L]$

$$\left\{1, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots, \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots\right\}$$

is generated by the set of complex exponentials $\{e^{\frac{in\pi x}{L}}\}_{n\in\mathbb{Z}}$. Consider the inner product for complex valued functions

$$\langle f, g\rangle = \int_{-L}^{L} f(x)\overline{g(x)}\, dx.$$

Since the complex conjugate $\overline{e^{\frac{in\pi x}{L}}} = e^{-\frac{in\pi x}{L}}$, it is also easy to check for $n \neq m$, that

$$\langle e^{\frac{in\pi x}{L}}, e^{\frac{im\pi x}{L}}\rangle = \int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{\frac{-im\pi x}{L}}\, dx = \frac{(-1)^{n-m} - (-1)^{m-n}}{i\pi(n-m)} = 0,$$

so the complex exponentials are an orthogonal set. We have the following reformulation of the full Fourier series using complex variables.

Definition 1. The *complex form of the full Fourier series* is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \tag{1}$$

where the (complex valued) Fourier coefficients are given by

$$c_n = \frac{\langle f(x), e^{\frac{in\pi x}{L}}\rangle}{\langle e^{\frac{in\pi x}{L}}, e^{\frac{in\pi x}{L}}\rangle} = \frac{\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\, dx}{\int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{in\pi x}{L}}\, dx} = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\, dx. \tag{2}$$

The proof of Parseval's equality also implies that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2L}\int_{-L}^{L} |f(x)|^2\, dx. \tag{3}$$

1.1.2 Fourier Transform

We now formally extend the Fourier series to the entire line by taking $L \to \infty$. If we substitute (2) into (1), then

$$f(x) = \frac{1}{2L}\sum_{n=-\infty}^{\infty}\left(\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\, dx\right)e^{\frac{in\pi x}{L}}.$$

We define $k_n = \frac{n\pi}{L}$ and $\Delta k = k_n - k_{n-1} = \frac{\pi}{L}$ then this simplifies to

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(x) e^{-ik_n x} \, dx \right) e^{ik_n x} \Delta k = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} C(k_n) e^{ik_n x} \Delta k.$$

where

$$C(k) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x) e^{-ikx} \, dx.$$

If we take $L \to \infty$, then

$$C(k) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x) e^{-ikx} \, dx \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \tag{4}$$

and interpreting the sum as a right Riemann sum,

$$f(x) = \lim_{L \to \infty} \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} C(k_n) e^{ik_n x} \Delta k \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(k) e^{ikx} dk. \tag{5}$$

Similarly, Parseval's equality (3) becomes

$$\int_{-L}^{L} |f(x)|^2 \, dx = 2L \sum_{n=-\infty}^{\infty} |c_n|^2 = 2L \sum_{n=-\infty}^{\infty} \frac{2\pi}{4L^2} \left| \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} \, dx \right|^2 = \sum_{n=-\infty}^{\infty} |C(k_n)|^2 \Delta k$$

so taking $L \to \infty$ implies

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |C(k)|^2 dk. \tag{6}$$

Remark 1. The step where we took $L \to \infty$ was not rigorous, because the bounds of integration and the function depend on L. A rigorous proof of this extension is much trickier.

1.2 Definition of the Fourier Transform

The Fourier transform $\mathcal{F}$ is an operator on the space of complex valued functions to complex valued functions. The coefficient $C(k)$ defined in (4) is called the Fourier transform.

Definition 2. Let $f : \mathbb{R} \to \mathbb{C}$. The *Fourier transform* of f is denoted by $\mathcal{F}[f] = \hat{f}$ where

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \tag{7}$$

Similarly, the *inverse Fourier transform* of f is denoted by $\mathcal{F}^{-1}[f] = \check{f}$ where

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx} \, dk. \tag{8}$$

The follows from (5) that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are indeed inverse operations.

Theorem 1 (*Fourier Inversion Formula*)

If f and f' are piecewise continuous, then $\mathcal{F}^{-1}[\mathcal{F}f] = f$ and $\mathcal{F}[\mathcal{F}^{-1}f] = f$. In particular,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk \quad \text{and} \quad f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{f}(x) e^{-ikx} \, dx.$$

Remark 2. Technically the Fourier inversion theorem holds for almost everywhere if f is discontinuous. In fact, one can show that $\mathcal{F}^{-1}[\mathcal{F}f] = \frac{f(x^-)+f(x^+)}{2}$, similarly to the pointwise convergence theorem.

The Fourier transforms of integrable and square integrable functions are also square integrable (6).

Theorem 2 (***Plancherel's Theorem***)

If f is both integrable and square integrable, then $\|f\|_{L^2} = \|\hat{f}\|_{L^2} = \|\check{f}\|_{L^2}$, i.e.

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk = \int_{-\infty}^{\infty} |\check{f}(x)|^2 dx. \tag{9}$$

Remark 3. In the Definition 2, we also assume that f is an integrable function, so that that its Fourier transform and inverse Fourier transforms are convergent.

Remark 4. Our choice of the symmetric normalization $\sqrt{2\pi}$ in the Fourier transform makes it a linear unitary operator from $L^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$, the space of square integrable functions $f : \mathbb{R} \to \mathbb{C}$. Different books use different normalizations conventions.

1.3 Properties of Fourier Transforms

The Fourier transform behaves very nicely under several operations of functions. We have already seen that the formulas for the solutions of several PDEs we have encountered can be expressed as convolutions.

Definition 3. The *convolution* of f and g is a function $f * g$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy = \int_{-\infty}^{\infty} f(y)g(x-y) \, dy. \tag{10}$$

We now state several properties satisfied by Fourier transforms.

Theorem 3 (***Properties of Fourier Transforms***)

If f and g are integrable, then

1. $\mathcal{F}[f(x-a)] = e^{-ika}\hat{f}(k)$
2. $\mathcal{F}[f(x)e^{ibx}] = \hat{f}(k-b)$
3. $\mathcal{F}[f(\lambda x)] = |\lambda|^{-1}\hat{f}(\lambda^{-1}k)$
4. $\mathcal{F}[\hat{f}(x)] = f(-k)$
5. $\mathcal{F}[f'(x)] = ik\hat{f}(k)$
6. $\mathcal{F}[xf(x)] = i\hat{f}'(k)$
7. $\mathcal{F}[(f * g)(x)] = \sqrt{2\pi}\hat{f}(k)\hat{g}(k)$
8. $\mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(k)$

Proof. We demonstrate how to prove some of these properties because the other proofs are similar.

(Property 1—4) These properties follow immediately from a change of variables. For example, property 1 follows because

$$\begin{aligned}
\mathcal{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)e^{-ikx} \, dx &\overset{y=x-a}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-ik(y+a)} \, dy \\
&= \frac{e^{-ika}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-iky} \, dy \\
&= e^{-ika}\hat{f}(k).
\end{aligned}$$

(Property 5—6) These properties follow immediately by interchanging differentiation and integration. For example, if we write $f(x)$ as the inverse Fourier transform of $\hat{f}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx} \, dk.$$

then computing the derivative implies that

$$f'(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ik\hat{f}(k) e^{ikx} \, dk = \mathcal{F}^{-1}[ikf(k)] \implies \mathcal{F}[f'(x)] = ik\hat{f}(k).$$

We are able to pass the derivative inside the integral by Leibiz's rule provided that $ik\hat{f}(k)e^{ikx}$ is integrable. This integrability condition is implicitly satisfied because we assumed that the functions in the theorem have convergent Fourier and inverse Fourier transforms.

(Property 7—8) These properties are the hardest to prove, so we will show it in detail. The proof of property 7 follows from a change of variables and Fubini's theorem

$$\begin{aligned}
\mathcal{F}[(f*g)(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x) e^{-ikx} \, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-ikx} \, dy dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) g(y) e^{-ikx} \, dx dy \quad \text{Fubini's Theorem} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z) g(y) e^{-ik(z+y)} \, dz dy \quad z = x - y \\
&= \sqrt{2\pi} \bigg(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{-ikz} \, dz \bigg) \bigg(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{-iky} \, dy \bigg) \\
&= \sqrt{2\pi} \hat{f}(k) \hat{g}(k).
\end{aligned}$$

To prove property 8, we can write $g(x)$ as the inverse Fourier transform of $\hat{g}$,

$$\begin{aligned}
\mathcal{F}[f(x)g(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \bigg(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\ell) e^{i\ell x} \, d\ell \bigg) e^{-ikx} \, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{g}(\ell) f(x) e^{-i(k-\ell)x} \, dx d\ell \quad \text{Fubini's Theorem} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\ell) \hat{f}(k-\ell) \, d\ell \quad \text{Definition (7)} \\
&= \frac{1}{\sqrt{2\pi}} (\hat{f} * \hat{g})(k) \quad \text{Definition (10)}
\end{aligned}$$

□

Remark 5. In Theorem 3, we also assume that f is nice enough so that the Fourier transforms and inverse Fourier transforms make sense. For example, in property 5 we need to assume that f is differentiable and the inverse Fourier transform of $ik\hat{f}(k)$ converges.

Remark 6. Property 5 can be proved by integration by parts if we assume that $f(\pm\infty) = 0$,

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}} f(x) e^{-ikx} \bigg|_{x=-\infty}^{x=\infty} + ik \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx = ik\hat{f}(k).$$

This is the standard proof of this property, and it requires that assumption that $f(\pm\infty) = 0$ for the boundary terms to vanish.

Remark 7. If we define $f_-(x) = f(-x)$, it is also useful to notice that a change of variables implies

$$\mathcal{F}^{-1}[f] = \check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx} \, dk \overset{k=-y}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-y) e^{-ixy} \, dy = \hat{f_-}(x) = \mathcal{F}[f_-]. \tag{11}$$

and

$$\mathcal{F}[f] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \overset{x=-y}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-y) e^{iky}\, dy = \check{f}_{-}(k) = \mathcal{F}^{-1}[f_{-}]. \tag{12}$$

Applying $\mathcal{F}$ or $\mathcal{F}^{-1}$ to these identities imply that

$$\mathcal{F}[\mathcal{F}[f(x)]] = f(-x) \iff \hat{\hat{f}}(x) = f(-x), \quad \mathcal{F}^{-1}[\mathcal{F}^{-1}[f(x)]] = f(-x) \iff \check{\check{f}}(x) = f(-x). \tag{13}$$

This observation is a generalization of property 4 in Theorem 3 and will be used many times to use the Fourier transform formulas "in reverse". For example, to prove property 8, we apply the inverse of property 7 by applying (12) and the first identity $f(-x) = \hat{\hat{f}}(x)$ in (13),

$$\mathcal{F}[f(x)g(x)] \overset{(12)}{=} \mathcal{F}^{-1}[f(-x)g(-x)] \overset{(13)}{=} \mathcal{F}^{-1}[\hat{\hat{f}}(x)\hat{\hat{g}}(x)] = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(k)$$

so

$$\mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(k).$$

1.4 Summary of the Properties of Fourier Transforms

1.4.1 Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

1.4.2 Inverse Fourier Transform

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx}\, dk$$

1.4.3 Plancherel's Theorem

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk = \int_{-\infty}^{\infty} |\check{f}(x)|^2 dx$$

1.4.4 Relationship Between Fourier Transforms and Inverse Fourier Transforms

1. $\mathcal{F}[\mathcal{F}^{-1}[f(x)]] = f(x)$
2. $\mathcal{F}^{-1}[\mathcal{F}[f(x)]] = f(x)$
3. $\mathcal{F}[\mathcal{F}[f(x)]] = f(-x)$
4. $\mathcal{F}^{-1}[\mathcal{F}^{-1}[f(x)]] = f(-x)$

1.4.5 List of Important Transformations

1. $\mathcal{F}[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2+k^2}\right)$
2. $\mathcal{F}[e^{-\frac{x^2}{2}}] = e^{-\frac{k^2}{2}}$

1.4.6 Properties of Fourier Transform

1. $\mathcal{F}[f(x-a)] = e^{-ika}\hat{f}(k)$
2. $\mathcal{F}[f(x)e^{ibx}] = \hat{f}(k-b)$
3. $\mathcal{F}[f(\lambda x)] = |\lambda|^{-1}\hat{f}(\lambda^{-1}k)$
4. $\mathcal{F}[\hat{f}(x)] = f(-k)$
5. $\mathcal{F}[f'(x)] = ik\hat{f}(k)$
6. $\mathcal{F}[xf(x)] = i\hat{f}'(k)$
7. $\mathcal{F}[(f * g)(x)] = \sqrt{2\pi}\hat{f}(k)\hat{g}(k)$
8. $\mathcal{F}[f(x)g(x)] = \frac{1}{\sqrt{2\pi}}(\hat{f} * \hat{g})(k)$

1.5 Finding Fourier Transforms

Problem 1.1. ($\star$) Find the Fourier transform of

$$f(x) = e^{-a|x|} \qquad a > 0.$$

Solution 1.1. This can be computed directly. We split the region of integration,

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-a|x|}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-ikx+ax}\, dx + \int_{0}^{\infty} e^{-ikx-ax}\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ikx+ax}}{a-ik} \Big|_{x=-\infty}^{x=0} + \frac{e^{-ikx-ax}}{-a-ik} \Big|_{x=0}^{x=\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-ik} + \frac{1}{a+ik} \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+k^2}.
\end{aligned}$$

Problem 1.2. ($\star\star$) Find the Fourier transform of

$$f(x) = e^{-\frac{x^2}{2}}.$$

Solution 1.2. This can be computed directly. We do a complex change of variables,

$$\begin{aligned}
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2}\, dx & \qquad \text{complete the square} \\
= \frac{1}{\sqrt{\pi}} e^{-\frac{k^2}{2}} \int_{-\infty}^{\infty} e^{-z^2}\, dz & \qquad \text{(see the Remark 9)} \\
= e^{-\frac{k^2}{2}}. & \qquad \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}.
\end{aligned}$$

Remark 8. There is also a proof of this fact that avoids complex analysis. If $f(x) = e^{-\frac{x^2}{2}}$, then

$$xf(x) = -f'(x).$$

Therefore, if we take the Fourier transform of both sides and apply Property 1.4.6.5 and 1.4.6.6 then

$$\mathcal{F}[xf(x)] = i\hat{f}'(k) = -ikf'(x) = \mathcal{F}[-f'(x)] \implies \hat{f}'(k) = -k\hat{f}(k).$$

Solving this ODE implies that

$$\hat{f}(k) = Ce^{-\frac{k^2}{2}}.$$

To find C, we plug in 0 and notice that

$$C = \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = 1,$$

so

$$\hat{f}(k) = e^{-\frac{k^2}{2}}.$$

Remark 9. The imaginary change of variables $z = \frac{1}{\sqrt{2}}(x+ik)$ can be justified using complex analysis,

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(x+ik)^2}\,dx = \sqrt{2}\int_{\mathbb{R}+\frac{ik}{\sqrt{2}}} e^{-z^2}\,dz.$$

For $k > 0$ consider the contour integral over the closed rectangular path oriented counter clockwise,

(1) $x + i\frac{k}{\sqrt{2}}$ for x from M to $-M$

(2) $-M + iy$ for y from $\frac{k}{\sqrt{2}}$ to 0

(3) x for x from $-M$ to M

(4) $M + iy$ for y from 0 to $\frac{k}{\sqrt{2}}$.

Since e^{-z^2} is analytic, the integral over this closed path is 0, so

$$\int_{(1)} e^{-z^2}\,dz + \int_{(2)} e^{-z^2}\,dz + \int_{(3)} e^{-z^2}\,dz + \int_{(4)} e^{-z^2}\,dz = 0.$$

Since e^{-z^2} is small when $\mathrm{Re}(z) = \pm M$, if we take $M \to \infty$, the integrals over the regions (2) and (4) vanish leaving us with

$$-\int_{\mathbb{R}+\frac{ik}{\sqrt{2}}} e^{-z^2}\,dz + \int_{-\infty}^{\infty} e^{-z^2}\,dz = 0 \implies \int_{\mathbb{R}+\frac{ik}{\sqrt{2}}} e^{-z^2}\,dz = \int_{-\infty}^{\infty} e^{-z^2}\,dz.$$

Therefore,

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(x+ik)^2}\,dx = \sqrt{2}\int_{\mathbb{R}+\frac{ik}{\sqrt{2}}} e^{-z^2}\,dz = \sqrt{2}\int_{-\infty}^{\infty} e^{-z^2}\,dz.$$

A similar contour can be defined when $k > 0$ (see Remark 15 for the definition of this contour).

Problem 1.3. ($\star$) Find the Fourier transform of

$$f(x) = xe^{-a\frac{x^2}{2}} \qquad a > 0.$$

Solution 1.3. Instead of computing it directly, we start from the Fourier transform of the Gaussian,

$$g(x) = e^{-\frac{x^2}{2}} \implies \hat{g}(k) = e^{-\frac{k^2}{2}}.$$

Since

$$f(x) = xe^{-a\frac{x^2}{2}} = x \cdot g(\sqrt{a}x),$$

the properties of the Fourier transform implies that

$$\begin{aligned}
\hat{f}(k) &= i\frac{d}{dk}\mathcal{F}[g(\sqrt{a}x)](k) && xf(x) \mapsto i\hat{f}'(k) \\
&= i\frac{d}{dk}\left(\frac{1}{\sqrt{a}}e^{-\frac{k^2}{2a}}\right) && f(\lambda x) \mapsto |\lambda|^{-1}\hat{f}(\lambda^{-1}k) \\
&= -ika^{-\frac{3}{2}}e^{-\frac{k^2}{2a}}.
\end{aligned}$$

Problem 1.4. ($\star$) Find the Fourier transform of

$$f(x) = (x^2 + a^2)^{-1} \sin(bx) \qquad a > 0,\, b > 0.$$

Solution 1.4. Instead of computing it directly, we start from the Fourier transform of the exponential,

$$g(x) = e^{-a|x|} \implies \hat{g}(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2 + k^2}\right).$$

Unfortunately, the inverse of this transformation is what appears in the problem. Using a change of variables (11) (or formula 1.4.4.3), we see that

$$\mathcal{F}[\mathcal{F}[g(s)]] = \mathcal{F}[\mathcal{F}^{-1}[g(-s)]] = g(-s),$$

so we can conclude that

$$h(x) = \hat{g}(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2 + x^2}\right) \implies \hat{h}(k) = \mathcal{F}[\mathcal{F}[g(k)]] = g(-k) = e^{-a|k|}.$$

Since

$$f(x) = (x^2 + a^2)^{-1} \sin(bx) = \frac{\sqrt{2\pi}}{2a} h(x) \sin(bx) = \frac{\sqrt{2\pi}}{2a}\left(\frac{h(x)e^{ibx} - h(x)e^{-ibx}}{2i}\right),$$

the properties of the Fourier transform implies that

$$\begin{aligned}\hat{f}(k) &= \frac{\sqrt{2\pi}}{4ai}\big(\hat{h}(k-b) - \hat{h}(k+b)\big) \qquad f(x)e^{ibx} \mapsto \hat{f}(k-b)\\ &= \frac{\sqrt{2\pi}}{4ai}\big(e^{-a|k-b|} - e^{-a|k+b|}\big).\end{aligned}$$

Remark 10. We have that the Fourier transform also satisfies the following properties

1. The Fourier transform of an integrable function is continuous.
2. The Fourier transform of a real valued odd function is a purely imaginary valued odd function.
3. The Fourier transform of a real valued even function is real valued even function.

We can use these facts as a sanity check for our answer. Notice that the Fourier transforms in Problems 1.1 and 1.2 are even in k and real valued. This is because the original functions were real valued even functions. Notice that the Fourier transforms in Problems 1.3 and 1.4 are odd in k and purely imaginary. This is because the original functions were real valued odd functions.

2 Solving PDEs on Infinite Domains

Fourier transforms can be used to convert PDEs on infinite domains into ODEs. This is a consequence of Property 1.4.6.5 applied inductively,

$$\mathcal{F}[f'(x)] = ik\hat{f}(k) \implies \mathcal{F}[f^{(n)}(x)] = ik\mathcal{F}[f^{(n-1)}(x)] = \ldots = (ik)^n \hat{f}(k). \tag{14}$$

This implies that the Fourier transforms can be used to "remove" partial derivatives in certain coordinates. We first explain the general procedure

1. Take the Fourier transform with respect to the unbounded variable. Solve the corresponding ODE of the transformed problem to get the general solution.
2. Use the transformed initial/boundary conditions to recover the particular solution of the transformed problem.
3. Use the inverse Fourier transform to recover the original solution.

Remark 11. The Fourier transform method to solve PDEs recovers solutions that decay sufficiently fast at $\pm\infty$ in the unbounded variable. This implicit assumption is required for the method to work, because we need the Fourier transforms and inverse Fourier transforms of the solution to be well-defined.

2.1 Example Problems

Problem 2.1. ($\star\star$) Using the properties of the Fourier transform, recover the formula for the bounded solution $u(x, y)$ of Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0 & x \in \mathbb{R},\ y > 0, \\ u|_{y=0} = g(x) & x \in \mathbb{R}. \end{cases} \tag{15}$$

Solution 2.1. We assume that $g(x)$ is integrable to ensure its Fourier transform exists.

Step 1 — Transform the Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to Laplace's equation, and consider its Fourier transform

$$\hat{u}(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ikx}\, dx.$$

Since $u_{xx} + u_{yy} = 0$, taking the Fourier transform of both sides implies that

$$-k^2 \hat{u}(k, y) + \hat{u}_{yy}(k, y) = 0 \qquad y > 0.$$

The solution to this ODE (in y) is given by

$$\hat{u}(k, y) = A(k)e^{-ky} + B(k)e^{ky},$$

where $A(k)$ and $B(k)$ are some yet to be determined functions of k.

Step 2 — Solve the Transformed Problem: Since our solution should decay at ∞ for $y \geq 0$ (if it wasn't, then its inverse Fourier transform won't exist), we have $B(k) = 0$ for $k > 0$ and $A(k) = 0$ for $k < 0$. The general solution can be simplified as

$$\hat{u}(k, y) = C(k)e^{-|k|y}, \qquad C(k) = \begin{cases} A(k) & k > 0 \\ A(0) + B(0) & k = 0 \\ B(k) & k < 0 \end{cases}$$

We can find $C(k)$ by using our initial condition,

$$u(x,0) = g(x) \implies C(k) = \hat{u}(k,0) = \hat{g}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,y) = \hat{g}(k)e^{-|k|y}.$$

Step 3 — Recover the Original Solution: We take the inverse Fourier transform of both sides to recover our original function. However, in order to get an answer that is not expressed in terms of a possibly complex valued Fourier transform of the initial condition g, we will do a bit more work to write our answer as a convolution. Let $S(x,y)$ be the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}e^{-|k|y}$,

$$\begin{aligned} S(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-y|k|}\,dk &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx+yk}\,dk + \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx-yk}\,dk \\ &= \frac{1}{2\pi}\frac{e^{ikx+yk}}{ix+y}\Big|_{k=-\infty}^{k=0} + \frac{1}{2\pi}\frac{e^{ikx-yk}}{ix-y}\Big|_{k=0}^{k=\infty} \\ &= \frac{1}{2\pi}\Big(\frac{1}{ix+y} - \frac{1}{ix-y}\Big) \\ &= \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}. \end{aligned}$$

Since $\hat{u}(k,y) = \hat{g}(k)e^{-|k|y} = \sqrt{2\pi}\hat{g}(k)\cdot\frac{e^{-|k|y}}{\sqrt{2\pi}}$, taking the inverse Fourier transform of both sides and applying Property 1.4.6.7 implies

$$u(x,y) = \mathcal{F}^{-1}\left[\sqrt{2\pi}\hat{g}(k)\cdot\frac{e^{-|k|y}}{\sqrt{2\pi}}\right] = (g(\cdot) * S(\cdot,y))(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\tau)^2+y^2}g(\tau)\,d\tau.$$

Remark 12. Instead of computing the Fourier transform directly, we could use property (11)

$$\mathcal{F}^{-1}[\mathcal{F}^{-1}[f(s)]] = \mathcal{F}^{-1}[\mathcal{F}[f(-s)]] = f(-s),$$

and the fact

$$\mathcal{F}[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{a^2+k^2}\right)$$

to conclude that (treating k as the variable and y as a constant)

$$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}e^{-|k|y}\right] = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left[\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\frac{2y}{y^2+x^2}\right]\right] = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}.$$

Remark 13. We need the boundedness assumption to get a unique solution. For example, if u solves Laplace's equation, then

$$v(x,y) = u(x,y) + Cy$$

also solves Laplace's equation (15) also solves (15) with the same boundary condition for any $C \in \mathbb{R}$.

Problem 2.2. ($\star\star$) Using the properties of the Fourier transform, recover the formula for the solution $u(x,t)$ of the heat equation

$$\begin{cases} u_t - u_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R}. \end{cases} \tag{16}$$

Solution 2.2. We assume that $g(x)$ is integrable to ensure its Fourier transform exists.

Step 1 — General Solution of Transformed Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to the heat equation, and consider its Fourier transform

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx.$$

Since $u_t - u_{xx} = 0$, taking the Fourier transform of both sides implies that

$$\hat{u}_t(k,t) + k^2 \hat{u}(k,t) = 0 \qquad t > 0.$$

The solution to this ODE (in t) is given by

$$\hat{u}(k,t) = A(k) e^{-k^2 t},$$

where $A(k)$ is some yet to be determined function of k.

Step 2 — Particular Solution of Transformed Problem: We can find $A(k)$ by using our initial condition,

$$u(x,0) = g(x) \implies A(k) = \hat{u}(k,0) = \hat{g}(k).$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,t) = \hat{g}(k) e^{-k^2 t}.$$

Step 3 — Original Solution: We take the inverse Fourier transform of both sides to recover our original function. However, in order to get an answer that is not expressed in terms of a possibly complex valued Fourier transform of the initial condition g, we will do a bit more work to write our answer as a convolution. Let $S(x,t)$ be the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}} e^{-k^2 t}$,

$$\begin{aligned} S(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-k^2 t}\, dk &= \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\, dk \\ &= \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \cdot \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-z^2}\, dz && \text{(See the Remark 15)} \\ &= \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}. && \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}. \end{aligned}$$

Since $\hat{u}(k,t) = \hat{g}(k) e^{-k^2 t} = \sqrt{2\pi} \hat{g}(k) \cdot \frac{e^{-k^2 t}}{\sqrt{2\pi}}$, taking the inverse Fourier transform of both sides and applying Property 1.4.6.7 implies

$$u(x,t) = \mathcal{F}^{-1}\left[\sqrt{2\pi} \hat{g}(k) \cdot \frac{e^{-k^2 t}}{\sqrt{2\pi}}\right] = (g(\cdot) * S(\cdot,t))(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\tau)^2}{4t}} g(\tau)\, d\tau.$$

This is the solution of the homogeneous heat equation proved in week 5.

Remark 14. Instead of computing the Fourier transform directly, we could use property (11) and the fact

$$\mathcal{F}[e^{-\frac{x}{2}^2}] = e^{\frac{-k^2}{2}},$$

to conclude that (treating k as the variable and t as a constant)

$$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}} e^{-k^2 t}\right] = \mathcal{F}\left[\frac{1}{\sqrt{2\pi}} e^{-k^2 t}\right] = \frac{1}{\sqrt{2\pi}} \mathcal{F}\left[e^{-\frac{(\sqrt{2t}k)^2}{2}}\right] = \frac{1}{\sqrt{4\pi t}} \cdot e^{-\frac{x^2}{4t}}.$$

Remark 15. The imaginary change of variables $z = \sqrt{t}k - i\frac{x}{2\sqrt{t}}$ can be justified using complex analysis,

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\, dk = \frac{1}{\sqrt{t}} \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\, dz.$$

Assuming $x > 0$, consider the contour integral over the closed rectangular path oriented counter clockwise,

(1) $k - i\frac{x}{2\sqrt{t}}$ for k from $-M$ to M

(2) $M + iy$ for y from $-\frac{x}{2\sqrt{t}}$ to 0

(3) k for k from M to $-M$

(4) $-M + iy$ for y from 0 to $-\frac{x}{2\sqrt{t}}$.

Since e^{-z^2} is analytic, the integral over this closed path is 0, so

$$\int_{(1)} e^{-z^2}\, dz + \int_{(2)} e^{-z^2}\, dz + \int_{(3)} e^{-z^2}\, dz + \int_{(4)} e^{-z^2}\, dz = 0.$$

Since e^{-z^2} is small when the $\mathrm{Re}(z) = \pm M$, if we take $M \to \infty$, the integrals over the regions (2) and (4) vanish leaving us with

$$\int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\, dz + \int_{\infty}^{-\infty} e^{-z^2}\, dz = 0 \implies \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\, dz = \int_{-\infty}^{\infty} e^{-z^2}\, dz.$$

Therefore,

$$\int_{\mathbb{R}} e^{-\left(\sqrt{t}k - i\frac{x}{2\sqrt{t}}\right)^2}\, dk = \frac{1}{\sqrt{t}} \int_{\mathbb{R} - i\frac{x}{2\sqrt{t}}} e^{-z^2}\, dz = \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-z^2}\, dz.$$

A similar contour can be defined when $x < 0$.

Problem 2.3. ($\star\star$) Using the properties of the Fourier transform, recover the formula for the solution $u(x,t)$ of the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = g(x) & x \in \mathbb{R} \\ u_t|_{t=0} = h(x) & x \in \mathbb{R}. \end{cases} \tag{17}$$

Solution 2.3. We assume that $g(x)$ and $h(x)$ are integrable to ensure its Fourier transform exists.

Step 1 — General Solution of Transformed Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to the wave equation, and consider its Fourier transform

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx.$$

Since $u_{tt} - u_{xx} = 0$, taking the Fourier transform of both sides implies that

$$\hat{u}_{tt}(k,t) + k^2 \hat{u}(k,t) = 0 \qquad t > 0.$$

The solution to this ODE (in t) is given by

$$\hat{u}(k,t) = A(k)\cos(kt) + B(k)\sin(kt),$$

where $A(k)$ and $B(k)$ is some yet to be determined function of k.

Step 2 — Particular Solution of Transformed Problem: We can find $A(k)$ and $B(k)$ by using our initial conditions,

$$u(x,0) = g(x) \implies A(k) = \hat{u}(k,0) = \hat{g}(k)$$

and

$$u_t(x,0) = h(x) \implies kB(k) = \hat{u}_t(k,0) = \hat{h}(k) \implies B(k) = \frac{\hat{h}(k)}{k}.$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,t) = \hat{g}(k)\cos(kt) + \frac{\hat{h}(k)}{k}\sin(kt).$$

Step 3 — Original Solution: We take the inverse Fourier transform of both sides to recover our original function. By linearity, it suffices to compute the inverse Fourier transforms of each term separately. The inverse Fourier transform of the first term follows immediately from 1.4.6.1

$$\mathcal{F}^{-1}[\hat{g}(k)\cos(kt)] = \mathcal{F}^{-1}\left[\hat{g}(k)\frac{e^{ikt}+e^{-ikt}}{2}\right] = \frac{g(x+t)+g(x-t)}{2}. \tag{18}$$

If we define $H(x) = \int_0^x h(s)\,ds$ then $H' = h$, so 1.4.6.5 implies that

$$\hat{h}(k) = \mathcal{F}[H'(x)] = ik\hat{H}(k) \implies \frac{\hat{h}(k)}{k} = i\hat{H}(k).$$

The inverse Fourier transform of the second term now follows immediately from 1.4.6.1

$$\mathcal{F}^{-1}\left[\frac{\hat{h}(k)}{k}\sin(kt)\right] = \mathcal{F}^{-1}\left[i\hat{H}(k)\frac{e^{ikt}-e^{-ikt}}{2i}\right] = \frac{H(x+t)-H(x-t)}{2} = \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds. \tag{19}$$

Therefore, using the linearity of the Fourier transforms, the computations (18) and (19) imply that

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{g}(k)\cos(kt) + \frac{\hat{h}(k)}{k}\sin(kt)\right] = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds.$$

This is d'Alembert's formula for the homogeneous wave equation proved in week 3.

Problem 2.4. ($\star$) Solve the following initial value problems

$$\begin{cases} u_t - u_{xx} - u = 0 & x \in \mathbb{R}, \quad t > 0, \\ u|_{t=0} = xe^{-\frac{x^2}{2}} & x \in \mathbb{R}. \end{cases} \tag{20}$$

You may leave your final answer as a real valued integral.

Solution 2.4. We will find a real valued integral representation of the solution $u(x,t)$.

Step 1 — General Solution of Transformed Problem: We take the Fourier Transform of our solution with respect to x. Let u be a solution to (20), and consider its Fourier transform

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx.$$

Since $u_t - u_{xx} - u = 0$, taking the Fourier transform of both sides implies that

$$\hat{u}_t(k,t) + k^2\hat{u}(k,t) - \hat{u}(k,t) = 0 \qquad t > 0.$$

The solution to this ODE (in t) is given by

$$\hat{u}(k,t) = A(k)e^{-(k^2-1)t},$$

where $A(k)$ is some yet to be determined function of k.

Step 2 — Particular Solution of Transformed Problem: We can find $A(k)$ by using our initial condition (see Problem 1.3)

$$u(x,0) = xe^{-\frac{x^2}{2}} \implies A(k) = \hat{u}(k,0) = \mathcal{F}[xe^{-\frac{x^2}{2}}](k) = -ike^{-\frac{k^2}{2}}.$$

Therefore, the Fourier transform of our solution is given by

$$\hat{u}(k,t) = -ike^{-\frac{k^2}{2}}e^{-(k^2-1)t} = -ike^{-(k^2-1)t-\frac{k^2}{2}}.$$

Step 3 — Original Solution: We take the inverse Fourier transform of both sides to recover our original function

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -ike^{-(k^2-1)t-\frac{k^2}{2}}e^{ikx}\,dk \\
&= -\frac{1}{\sqrt{2\pi}}\int_0^{\infty} ike^{-(k^2-1)t-\frac{k^2}{2}}e^{ikx}\,dk - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} ike^{-(k^2-1)t-\frac{k^2}{2}}e^{ikx}\,dk \\
(\tilde{k} = -k) \quad &= -\frac{1}{\sqrt{2\pi}}\int_0^{\infty} ike^{-(k^2-1)t-\frac{k^2}{2}}e^{ikx}\,dk + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} i\tilde{k}e^{-(\tilde{k}^2-1)t-\frac{\tilde{k}^2}{2}}e^{-i\tilde{k}x}\,d\tilde{k} \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} ke^{-(k^2-1)t-\frac{k^2}{2}}\frac{e^{ikx}-e^{-ikx}}{2i}\,dk \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} ke^{-(k^2-1)t-\frac{k^2}{2}}\sin(kx)\,dk.
\end{aligned}$$

This is a real valued integral, so we can stop here.

Remark 16. Notice that this computation to express the answer in terms of a real valued integral is equivalent to taking the real part of our solution,

$$\begin{aligned}
u(x,t) &= \mathrm{Re}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -ike^{-(k^2-1)t-\frac{k^2}{2}}e^{ikx}\,dk\right] \\
&= \mathrm{Re}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -ike^{-(k^2-1)t-\frac{k^2}{2}}(\cos(kx)+i\sin(kx))\,dk\right] \qquad e^{i\theta} = \cos(\theta) + i\sin(\theta) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ke^{-(k^2-1)t-\frac{k^2}{2}}\sin(kx)\,dk \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} ke^{-(k^2-1)t-\frac{k^2}{2}}\sin(kx)\,dk. \qquad \text{the integrand is even}
\end{aligned}$$

Remark 17. In contrast to Problems 2.1 and 2.2, we have an explicit form of the initial condition so we simply took the inverse Fourier transform in the last step. We went through the extra work in Problems 2.1 and 2.2 to write in terms of a convolution to avoid leaving our answer in terms of a (possibly complex) Fourier transform in the initial condition.