1 Properties of Laplace's Equation in $\mathbb{R}^2$

Consider an open set $\Omega \subseteq \mathbb{R}^2$. Solutions to Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega$$

are called *harmonic functions*. Harmonic functions in $\mathbb{R}^2$ are closely related to analytic functions in complex analysis. We discuss several properties related to Harmonic functions from a PDE perspective.

We first state a fundamental consequence of the divergence theorem (also called the divergence form of Green's theorem in 2 dimensions) that will allow us to simplify the integrals throughout this section.

Definition 1. Let Ω be a bounded open subset in $\mathbb{R}^2$ with smooth boundary. For $u, v \in C^2(\bar{\Omega})$, we have

$$\iint_\Omega \nabla v \cdot \nabla u \, dxdy + \iint_\Omega v \Delta u \, dxdy = \oint_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds. \tag{1}$$

where $n = n(x, y)$ is the outward pointing unit normal at $(x, y) \in \partial\Omega$ and $\frac{\partial u}{\partial n} := \nabla u \cdot n$ is the corresponding outward normal derivative of u. This identity is called *Green's first identity*.

Remark 1. To simplify notation, we restricted the analysis to $\mathbb{R}^2$ but all results in this section generalize easily to $\mathbb{R}^n$. The domains Ω is always assumed to have smooth boundary.

1.1 Mean Value Property

We will show that the values of harmonic functions is equal to the average over balls of the form

$$B_r(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} \le r\} \subset \Omega.$$

Theorem 1 (*Mean Value Property*)

If $u \in C^2(\Omega)$ is harmonic in Ω, then

$$u(x_0, y_0) = \frac{1}{2\pi r} \oint_{\partial B_r(x_0,y_0)} u \, ds = \frac{1}{\pi r^2} \iint_{B_r(x_0,y_0)} u \, dxdy \tag{2}$$

for any ball $B_r(x_0, y_0) \subset \Omega$.

Proof. We fix $(x_0, y_0) \in \Omega$. We begin by showing the first equality in (2).

First Equality: Consider the function

$$f(r) = \frac{1}{2\pi r} \oint_{\partial B_r(x_0,y_0)} u \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) \, d\theta$$

using the counter clockwise parametrization of $\partial B_r(x_0, y_0)$ with $x(\theta) = x_0 + r\cos(\theta)$ and $y(\theta) = x_0 + r\sin(\theta)$ for $-\pi \le \theta \le \pi$. Differentiating with respect to r implies that

$$\begin{aligned}
f'(r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\theta) u_x(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) + \sin(\theta) u_y(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) \, d\theta \\
&= \frac{1}{2\pi r} \oint_{\partial B_r(x_0,y_0)} \frac{x - x_0}{r} u_x(x, y) + \frac{y - y_0}{r} u_y(x, y) \, ds \\
&= \frac{1}{2\pi r} \oint_{\partial B_r(x_0,y_0)} \frac{\partial u}{\partial n} \, ds \qquad n = \frac{1}{r}(x - x_0, y - y_0) \\
&= \frac{1}{2\pi r} \iint_{B_r(x_0,y_0)} \Delta u \, dxdy \quad \text{Green's first identity (1)} \\
&= 0
\end{aligned} \tag{3}$$

since $\Delta u = 0$ on $B_r(x_0,y_0) \subset \Omega$. Therefore $f(r)$ is constant. To figure out the value of $f(r)$, we take the limit as $r \to 0$ and apply continuity to see that

$$\lim_{r\to 0} f(r) = \lim_{r\to 0} \frac{1}{2\pi r} \oint_{\partial B_r(x_0,y_0)} u\, ds = \lim_{r\to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, d\theta = u(x_0,y_0).$$

Second Equality: From the previous section, for all $\rho \le r$ we have

$$u(x_0,y_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta))\, d\theta.$$

We can multiply both sides by ρ and integrate to conclude that

$$\int_0^r \rho u\, d\rho = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^r u(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta))\rho\, d\rho d\theta.$$

The term on the left simplifies to $\frac{r^2}{2}u(x,y)$ and the term on the right is the integral over $B_r(x_0,y_0)$ expressed in polar coodinates, so

$$u(x,y) = \frac{1}{\pi r^2} \int_{-\pi}^{\pi} \int_0^r u(x_0 + \rho\cos(\theta), y_0 + \rho\sin(\theta))\rho\, d\rho d\theta = \frac{1}{\pi r^2} \iint_{B_r(x_0,y_0)} u\, dxdy.$$

□

Remark 2. The normalizations appearing in (2) are the circumference of a circle and the area of a disc. This normalization means that the integrals can be interpreted as the expected value of u over a uniform probability measure on the circle and disc.

The converse of Theorem 1 is also true, so the mean value property characterizes harmonic functions.

Theorem 2 (*Converse of the Mean Value Property*)

If $u \in C^2(\Omega)$ satisfies (2) for every ball $B_r(x_0,y_0) \subset \Omega$, then u is harmonic in Ω.

Proof. Suppose that $\Delta u \not\equiv 0$ in Ω. Without loss of generality, suppose that $\Delta u(x_0,y_0) > 0$ at some (x_0,y_0) in Ω. By continuity, there exists a ball $B_\rho(x_0,y_0)$ such that $\Delta u > 0$ within $B_\rho(x_0,y_0)$. For $r \le \rho$, consider

$$f(r) = \frac{1}{2\pi r} \oint_{\partial B_r(x_0,y_0)} u\, ds.$$

If u satisfies (2) for every ball in the domain, then clearly f must be constant since it must equal $u(x_0,y_0)$ for all $r \le \rho$. The computations leading to (3) implies that

$$f'(r) = \frac{1}{2\pi r} \iint_{B_r(x_0,y_0)} \Delta u\, dxdy > 0,$$

which contradicts the fact that $f(r)$ must be constant. If $\Delta u < 0$ within some $B_\rho(x_0,y_0)$, then we arrive at the same contradiction because the f will be strictly decreasing in that scenario. □

Remark 3. Theorem 1 and Theorem 2 implies that u is harmonic if and only if it satisfies the mean value property. This characterization of harmonic functions is also valid in $\mathbb{R}^n$.

1.2 Maximum Principle

Harmonic functions also attain its extreme values on the boundary of the set. This implies that the maximum/minimum of solutions to $\Delta u = 0$ are determined by the boundary conditions.

Theorem 3 (***Weak Maximum Principle***)

Let Ω be a connected bounded open set $\Omega \subseteq \mathbb{R}^n$. If u is harmonic in Ω and u is continuous on $\bar{\Omega}$, then the maximum and minimum values of u are attained on $\partial\Omega$.

Proof. The proof is identical to the maximum principle for the heat equation proof. Let u be a harmonic on Ω and continuous on $\bar{\Omega}$. Let $\epsilon > 0$ and define $v^\epsilon(x, y) = u(x, y) + \epsilon(x^2 + y^2)$. The following interior point condition holds

$$v^\epsilon_{xx} + v^\epsilon_{yy} = v^\epsilon_{xx} + v^\epsilon_{yy} + 4\epsilon \geq 0 + 4\epsilon > 0 \text{ in } \Omega.$$

By the second derivative test, any interior maximum must satisfy the critical point condition $v^\epsilon_{xx}+v^\epsilon_{yy} \leq 0$, which is impossible because it contradicts the interior point condition $v^\epsilon_{xx} + v^\epsilon_{yy} > 0$. Therefore, $v^\epsilon(x, y)$ does not attain an interior maximum.

Since $v^\epsilon(x, y)$ is a continuous function and $\bar{\Omega}$ is compact, v^ϵ attains a maximum at some point $(\tilde{x}, \tilde{y}) \in \partial\Omega$. We are on a bounded domain, $x^2 + y^2 \leq M$ for all $(x, y) \in \partial\Omega$. Since $0 \leq \epsilon(x^2 + y^2) \leq M\epsilon$, we have

$$\max_{\Omega} u(x, y) \leq \max_{\Omega} v^\epsilon(x, y) \leq v^\epsilon(\tilde{x}, \tilde{y}) \leq u(\tilde{x}, \tilde{y}) + \epsilon(\tilde{x}^2 + \tilde{y}^2) \leq \max_{\partial\Omega} u(x, y) + 2M.$$

The upperbound holds for all $\epsilon > 0$, so taking $\epsilon \to 0$ implies

$$\max_{\Omega} u(x, y) \leq \max_{\partial\Omega} u(x, y) \implies \max_{\Omega} u(x, y) = \max_{\partial\Omega} u(x, y)$$

as required. The proof of the minimum principle follows by applying the maximum principle to $-u(x, y)$ and using the fact $\Delta(-u) = 0$. □

Remark 4. The weak maximum principle can be used to prove uniqueness and stability of continuous solutions to (4). These proofs are similar to the proofs in the setting of the heat equation.

1.3 Uniqueness

We now provide two proofs of uniqueness of Poisson's equation with Dirichlet boundary conditions,

$$\begin{cases} \Delta u = f(x, y) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = g(x, y) & \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

Theorem 4 (***Uniqueness of the Dirichlet Problem***)

Continuous solutions to (4) are unique.

Proof Using the Maximum Principle. Consider two continuous solutions u and v of the Dirichlet problem (4). It is easy to see that $w = u - v$ solves

$$\begin{cases} \Delta w = 0 & \text{in} \quad \Omega, \\ w|_{\partial\Omega} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The function w is continuous and harmonic so it satisfies the maximum principle. Since $w = 0$ on $\partial\Omega$, for any $(x, y) \in \Omega$

$$0 = \min_{\partial\Omega} w \leq w(x, y) \leq \max_{\partial\Omega} w = 0$$

so $w = 0$ in Ω as well. Therefore, $w = 0$ in $\bar{\Omega}$, so $u = v$. □

We now present a proof using the energy method. Given a function $w \in C^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open domain, we define energy of w as

$$E[w] = \iint_\Omega |\nabla w|^2 \, dxdy.$$

Our goal is to show that $E[w] \leq 0$ which will imply $E[w] = 0$. This will prove that $\nabla w = 0$ in Ω, which proves that w is constant on the interior. If we can show that $w = 0$ on $\partial\Omega$, then we can prove uniqueness of continuous solutions.

Proof Using the Energy Method. Consider two continuous solutions u and v of the Dirichlet problem (4). It is easy to see that $w = u - v$ solves

$$\begin{cases} \Delta w = 0 & \text{in} \quad \Omega, \\ w|_{\partial\Omega} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Computing the energy of w, we have

$$\begin{aligned} E[w] &= \iint_\Omega |\nabla w|^2 \, dxdy \\ &= \oint_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds - \iint_\Omega w \Delta w \, dxdy && \text{Green's First Identity} \\ &= \oint_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds && \Delta w = 0 \\ &= \oint_{\partial\Omega} 0 \, ds = 0 && \text{Boundary Conditions.} \end{aligned}$$

Since $E[w] \geq 0$ the computation above implies $0 \leq E[w] \leq 0$, so $E[w] = 0$. The integrand is non-negative, so by continuity,

$$\nabla w \equiv 0 \text{ in } \Omega \quad \Longrightarrow \quad w = \text{constant in } \bar{\Omega}.$$

Since $w = 0$ on $\partial\Omega$, we must have $w \equiv 0$ which implies $u = v$ on $\bar{\Omega}$. □

Remark 5. The energy argument can be adapted to problems on the exterior of a circle using a change of variables called the *Kelvin transform*, $\tilde{u}(x, y) = u(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. In polar coordinates, this transformation is $\tilde{u}(r, \theta) = u(r^{-1}, \theta)$, so it is straightforward to check that u is harmonic if and only if $\bar{u}$ is harmonic. The domain of $\bar{u}$ is a circle if u is defined on the exterior of a circle, so we can apply the energy argument to $\tilde{u}$ when u is defined on the exterior of a circle.

1.4 Poisson's Formula

Recall that the solution to Laplace's equation on the interior of the disk $B_a(0,0)$ (see the computation in Week 10 Summary Problem 1.1)

$$\begin{cases} \Delta u = 0 & r < a, \quad -\pi \leq \theta \leq \pi, \\ u|_{r=a} = g(\theta), \end{cases} \tag{5}$$

is given by

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

where

$$A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} g(\phi) \cos(n\phi) \, d\phi \quad \text{and} \quad B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} g(\phi) \sin(n\phi) \, d\phi.$$

In this example, the Fourier series is summable, so we can get a closed form representation for u. Replacing the Fourier coefficients with the integrals, we see that

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)\,d\phi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_{-\pi}^{\pi} g(\phi)(\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta))\,d\phi$$

By the product sum identities, this simplifies to

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi))\right) d\phi.$$

The infinite series is geometric, so it can be explicitly computed to give

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi)) &= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{ni(\theta-\phi)} + \left(\frac{r}{a}\right)^n e^{-ni(\theta-\phi)} \\ &= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}} \qquad \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \\ &= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}. \end{aligned}$$

Theorem 5 (*Poisson's Formula*)

The solution to (5) is given by

$$u(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_{-\pi}^{\pi}\frac{g(\phi)}{a^2 - 2ar\cos(\theta-\phi)+r^2}\,d\phi. \tag{6}$$

Poisson's formula also has an alternate representation in Cartesian coordinates. For $r < a$, let

$$(x,y) = (r\cos(\theta), r\sin(\theta)) \in B_a^\circ(0,0) \quad \text{and} \quad (\tilde{x},\tilde{y}) = (a\cos(\phi), a\sin(\phi)) \in \partial B_a(0,0).$$

The cosine law implies that

$$(x-\tilde{x})^2 + (y-\tilde{y})^2 = a^2 - 2ar\cos(\theta-\phi) + r^2.$$

In polar coordinates, $\tilde{x} = a\cos(\phi)$ and $\tilde{y} = a\sin(\phi)$ so $ad\phi = d\tilde{s}$, which implies that (6) simplifies to

$$u(x,y) = \frac{a^2 - (x^2+y^2)}{2\pi a}\oint_{\partial B_a(0,0)}\frac{u(\tilde{x},\tilde{y})}{(x-\tilde{x})^2 + (y-\tilde{y})^2}\,d\tilde{s}. \tag{7}$$

As an application of this formula, we show that harmonic functions are smooth.

Theorem 6 (*Regularity of Harmonic Functions*)

If u is harmonic in $\Omega \subseteq \mathbb{R}^2$, then $u \in C^\infty(\Omega)$.

Proof. Let $(x_0,y_0) \in \Omega$ and suppose that $r > 0$ is chosen sufficiently small so that $B_r(x_0,y_0) \subset \Omega$. After recentering (7), we can conclude for $(x,y) \in B_r^\circ(x_0,y_0) \subset \Omega$ that

$$u(x,y) = \frac{r^2 - (x-x_0)^2 - (y-y_0)^2}{2\pi r}\oint_{\partial B_r(x_0,y_0)}\frac{u(\tilde{x},\tilde{y})}{(x-\tilde{x})^2+(y-\tilde{y})^2}\,d\tilde{s}. \tag{8}$$

Since $(x,y) \in B_r^\circ(x_0,y_0)$, there exists a m such that $(x-\tilde{x})^2 + (y-\tilde{y})^2 > m$ on the domain of integration. The integrand is infinitely differentiable for all $x,y \in B_r(x_0,y_0)$, so u is smooth in Ω. $\square$

Remark 6. Poisson's formula (8) also gives an alternate proof of the mean value property (Theorem 1),

$$u(x,y)\Big|_{(x,y)=(x_0,y_0)} = \frac{r^2}{2\pi r}\oint_{\partial B_r(x_0,y_0)}\frac{u(\tilde{x},\tilde{y})}{r^2}\,d\tilde{s} = \frac{1}{2\pi r}\oint_{\partial B_r(x_0,y_0)} u(\tilde{x},\tilde{y})\,d\tilde{s}.$$

1.5 Example Problems

Problem 1.1. ($\star$) Find the maximimum and minimum values of $u = x^3 - 3xy^2 + 3x$ on the disk $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$.

Solution 1.1. We have

$$u_{xx} + u_{yy} = 6x - 6x = 0$$

so u is harmonic in $\mathbb{R}^2$. By the weak maximum principle (Theorem 3) u attains its maximum on the circle $\partial\Omega = \{(x, y) : x^2 + y^2 = 1\}$. In polar coordinates, we have

$$u(1, \theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + 3\cos(\theta) =: f(\theta).$$

We have

$$\begin{aligned} f'(\theta) &= -3\cos^2(\theta)\sin(\theta) - 6\cos^2(\theta)\sin(\theta) + 3\sin^3(\theta) - 3\sin(\theta) \\ &= 3\sin(\theta)(\sin^2\theta - 3\cos^2(\theta) - 1) \\ &= -12\sin(\theta)\cos^2(\theta) \end{aligned}$$

which is equal to 0 when $\theta = 0, \pm\pi, \pm\frac{\pi}{2}$ on the domain $[-\pi, \pi]$. Since

$$f(0) = 4 \quad f(\pm\pi) = -4 \quad f\Big(\pm\frac{\pi}{2}\Big) = 0$$

the maximum value is 4 and the minimum value is -4.

Problem 1.2. ($\star$) Suppose u is harmonic in the disc $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and continuous on $\bar{\Omega}$. If u is equal to x^2y^2 on $\partial\Omega$, find $u(0, 0)$.

Solution 1.2. By the mean value property (Theorem 1), we have

$$\begin{aligned} u(0,0) = \frac{1}{2\pi}\oint_{\partial\Omega} u\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\cos(\theta), \sin(\theta))\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(\theta)\sin^2(\theta)\,d\theta \\ &= \frac{1}{8\pi}\int_{-\pi}^{\pi} \sin^2(2\theta)\,d\theta \\ &= \frac{1}{8}. \end{aligned}$$