# 1 Class properties of a CTMC

We now explain the generalization of the class properties for DTMC to the continuous time case. In general, if the properties depend on time, then there are some slight differences with the discrete case.

## 1.1 Classification of the states of a CTMC

The classification of the states of a CTMC is done in the same way as for a DTMC.

**Definition 1.1.**

1. A state $j$ is called **accessible** from state $i$ if there exists $t \geq 0$ such that
$$p_{ij}(t) > 0.$$
2. We say state $i$ and $j$ **communicate**, if there exists $t, s \geq 0$ such that
$$p_{ij}(t) > 0 \text{ and } p_{ji}(s) > 0.$$
3. A CTMC is called **irreducible** if all states communicate with each other.

Clearly, the relation of communication is still an equivalence relation.

**Remark 1.2.** For a CTMC, there is no concept of "periodicity". This is because the sojourn times are random (exponentially distributed).

We define
$$T_i = \inf\{t > 0 : X(t) = i \text{ and } X(s) \neq i \text{ for some } s \in (0,t)\}$$
as the first time of a return to state $i$. Then we let
$$\mu_i = \mathbb{E}[T_i | X(0) = i]$$
be the mean of the first time of a return to state $i$. The concepts of recurrence and null recurrence can now be extended to continuous time.

**Definition 1.3.**

1. A state $i$ is called **transient** if $\mathbb{P}(T_i < \infty | X(0) = i) < 1$.
2. A state $i$ is called **recurrent** if $\mathbb{P}(T_i < \infty | X(0) = i) = 1$.

**Definition 1.4.**

1. A recurrent state $i$ is called **positive recurrent** if $\mu_i < \infty$,
2. A recurrent state $i$ is called **null recurrent** if $\mu_i = \infty$.

Just as with the DTMC, we have that recurrence and positive recurrence are class properties.

**Proposition 1.5**

*Recurrence/transience and positive/null recurrence are class properties, i.e., states in the same communication class all share the same properties.*

If our state space is finite, then the expected time to return to every state is finite.

**Proposition 1.6**

*An irreducible CTMC with a **finite state space** is positive recurrent.*

Several of the class properties can be obtained by looking at the behavior of the embedded DTMC.

**Proposition 1.7**

*(1) A CTMC is irreducible if and only if its embedded DTMC is irreducible.*

*(2) A state $i$ is recurrent/transient for a CTMC if and only if it is recurrent/transient for the embedded DTMC.*

**Remark 1.8.** Notice that a CTMC and its embedded DTMC do **NOT** necessarily have the same positive/null recurrence properties

## 1.2 Stationary distribution and limiting distribution

The concept of the stationary distribution and limiting distribution are identical as in the discrete time case, if we replace the $n$ with $t$.

**Definition 1.9.** A vector $\boldsymbol{\pi}$ is called a **stationary distribution** for a CTMC with transition matrix $P(t)$ if

1. $\pi_i \geq 0$ for all $i \in S$ and $\sum_{i \in S} \pi_i = 1$ .
2. $\boldsymbol{\pi}\boldsymbol{P}(t) = \boldsymbol{\pi}$ for all $t \geq 0$.

**Remark 1.10.** Let $\boldsymbol{\nu}(\mathbf{t}) = (\nu_i(t))_{i \in S}$ be the distribution of $X(t)$, that is

$$\nu_i(t) = \mathbb{P}\left(X(t) = i\right).$$

Clearly, we still have

$$\begin{aligned}\nu_i(t) &= \sum_{k \in S} \mathbb{P}\left( X(t) = i \middle| X(0) = k\right) \mathbb{P}\left(X(0) = k\right) \\ &= \sum_{k \in S} \nu_k(0) p_{ki}(t) \\ &= (\boldsymbol{\nu}(0)\boldsymbol{P}(t))_i.\end{aligned}$$

If a stationary distribution $\boldsymbol{\pi}$ exists and we take $\boldsymbol{\nu}(0) = \boldsymbol{\pi}$ as starting distribution, we have $\boldsymbol{\nu}(t) = \boldsymbol{\pi}$ for all $t \geq 0$.

The stationary distribution can be found as usual by solving the relevant system of equations. However, we see that instead of solving it for all $t$, the next theorem gives a characterization of stationary distributions by means of the $Q$-matrix (the infinitesimal generator).

**Theorem 1.11**

$\boldsymbol{\pi}\boldsymbol{P}(t) = \boldsymbol{\pi}$ *for any* $t \geq 0$ *if and only if*

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}.$$

Therefore, to find stationary distribution(s), it is sometimes easier to use the conditions

$$\begin{cases} \boldsymbol{\pi}\boldsymbol{Q} = \mathbf{0} \\ \pi \cdot 1 = 1 \\ \pi_i \geq 0 \text{ for any } i \in S. \end{cases}$$

As with the DTMC, we have a characterization of the limiting distribution in terms of the recurrence times and the stationary distribution.

**Theorem 1.12** (***Fundamental limit theorem***)

*Consider an* ***irreducible*** *CTMC. We have the following:*

1. *If the CTMC is* ***positive recurrent****, a unique stationary distribution and limiting distribution exists and is given by*

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j = \frac{\alpha_j^{-1}}{\mu_j}, \qquad \text{for any } i, j \in S.$$

2. *If the CTMC is* ***transient or null recurrent****, a stationary distribution does not exist and*

$$\lim_{t\to\infty} p_{ij}(t) = 0, \qquad \text{for any } i, j \in S.$$

**Remark 1.13.** Like DTMC, for an irreducible and positive recurrent CTMC, the stationary distribution $\pi_j$ also represents the long-run proportion of time that the CTMC spends in state $j$. That is,

$$\text{the long-run proportion of time in } j = \frac{\text{expected amount of time in } j \text{ during a circle of revisiting}}{\text{expected circle length of revisiting}}.$$

This explains the $\alpha_j^{-1}$ factor in the numerator.

## 1.3 Example Problems

### 1.3.1 Proofs of Main Results

**Problem 1.1.** Prove Theorem 1.11.

**Proof.** Recall that $\boldsymbol{P}(t) = e^{t\boldsymbol{Q}} = \sum_{n=0}^{\infty} \frac{t^n \boldsymbol{Q}^n}{n!}$ and $\boldsymbol{Q}^0 = \boldsymbol{I}$, we have

$$\begin{aligned} \boldsymbol{\pi Q} = \boldsymbol{0} &\iff \boldsymbol{\pi Q}^n = \boldsymbol{0} \text{ for any } n \geq 1 \\ &\iff \boldsymbol{0} = \sum_{n=1}^{\infty} \frac{t^n}{n!} \boldsymbol{\pi Q}^n = \sum_{n=1}^{\infty} \frac{t^n}{n!} \boldsymbol{\pi Q}^n + \frac{t^0}{0!} \boldsymbol{\pi Q}^0 - \boldsymbol{\pi} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \pi \boldsymbol{Q}^n - \boldsymbol{\pi}, \quad \text{for any } t \geq 0 \\ &\iff \boldsymbol{\pi P}(t) = \boldsymbol{\pi}, \qquad \text{for any } t \geq 0. \end{aligned}$$

□

### 1.3.2 Applications

**Problem 1.2.** Consider a two-state CTMC with transition matrix $\alpha, \beta > 0$.

$$\boldsymbol{P}(t) = \begin{pmatrix} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t} & \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t} \\ \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t} & \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t} \end{pmatrix}$$

What is the stationary distribution? And what are the mean recurrence times for states $i$ and $j$.

**Solution 1.1.** This is a irreducible and it has a finite state space, so it positive recurrent CTMC. We can apply the fundamental limit theorem. We first find the stationary distribution by applying Theorem 1.11. We have that

$$\boldsymbol{Q} = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}.$$

One can easily check that $\boldsymbol{\pi} = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$ satisfies $\boldsymbol{\pi Q} = \mathbf{0}$. Furthermore, we have that

$$\frac{\alpha_0^{-1}}{\mu_0} = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha}+\frac{1}{\beta}} = \frac{\beta}{\alpha+\beta} = \pi_0 \implies \mu_0 = \frac{\alpha_0^{-1}}{\pi_0},$$
$$\frac{\alpha_1^{-1}}{\mu_1} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha}+\frac{1}{\beta}} = \frac{\alpha}{\alpha+\beta} = \pi_1 \implies \mu_1 = \frac{\alpha_1^{-1}}{\mu_1}.$$

**Remark 1.14.** We can actually solve this problem by hand, which gives some intuition on the statements of fundamental limit theorem. Our formula for $\boldsymbol{P}(t)$ immediately yields the limiting distribution:

$$\lim_{t\to\infty} \boldsymbol{P}(t) = \lim_{t\to\infty} \begin{pmatrix} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}e^{-(\alpha+\beta)t} & \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}e^{-(\alpha+\beta)t} \\ \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}e^{-(\alpha+\beta)t} & \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}e^{-(\alpha+\beta)t} \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}.$$

The rows of the limiting matrix gives the stationary distribution $\boldsymbol{\pi} = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$ as before.

To compute the mean recurrence time we can use the properties of exponential random variables. Note that the sojourn time $U_0 \sim$ EXP with parameter $\alpha_0 = -q_{00} = \alpha$, and the sojourn time $U_1 \sim$ EXP with parameter $\alpha_1 = -q_1 = \beta$. Given the CTMC starts from $i = 0, 1$, the first times of return to state $i$ is

$$U_i + U_j = U_0 + U_1, \qquad j \neq i$$

Therefore,

$$\mu_0 = \mathbb{E}[T_0|X(0) = 0] = \mathbb{E}[U_0] + \mathbb{E}[U_1] = \frac{1}{\alpha} + \frac{1}{\beta} = \mathbb{E}[U_1] + \mathbb{E}[U_0] = \mathbb{E}[T_1|X(0) = 1] = \mu_1,$$

which shows that

$$\frac{\alpha_0^{-1}}{\mu_0} = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha}+\frac{1}{\beta}} = \frac{\beta}{\alpha+\beta} = \pi_0,$$
$$\frac{\alpha_1^{-1}}{\mu_1} = \frac{\frac{1}{\beta}}{\frac{1}{\alpha}+\frac{1}{\beta}} = \frac{\alpha}{\alpha+\beta} = \pi_1.$$