# 1 Calculus Review

## 1.1 Change of Variables in Multiple Dimensions

The change of variables formula allows us to change the dummy variable of integration, potentially simplifying the integrals we compute. The class of nice functions for which we are able to do a change of variables with are called diffeomorphisms.

**Definition 1.1.** A **diffeomorphism** is a map $F : U \to V$ of $\mathbb{R}^n$ such that $F$ is a bijection and both $F$ and $F^{-1}$ are differentiable.

**Proposition 1.2 (*Substitution Rule*)**

*Let $x = g(u)$. If this map is a diffeomorphism then*

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,du.$$

We can remember this change of variables by using the fact that

$$x = g(u) \implies \frac{dx}{du} = g'(u) \implies dx = g'(u)\,du, \qquad a \le x \le b \to g^{-1}(a) \le u \le g^{-1}(b)$$

and substituting the appropriate parts into the integration formula.

The same trick works in higher dimensions, the key difference being that $g'(x)$ will be replaced by the Jacobian matrix.

**Definition 1.3.** The **Jacobian** of the transformation $x = g(u, v)$, $y = h(u, v)$ is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

where $|\cdot|$ denotes the determinant of a matrix.

**Proposition 1.4 (*Change of Variables*)**

*Let $x = g(u, v)$ and $y = h(u, v)$. If this map is a diffeomorphism then*

$$\iint_R f(x, y)\,dxdy = \iint_S f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv.$$

*where $R$ is the image of $S$ under the map $x = g(u, v)$ and $y = h(u, v)$.*

## 1.2 Example Problems

**Problem 1.1.** Compute the double integral $\iint_D r \cot^n \theta drd\theta$ over the region $D = [1, a] \times [1, a]$ in the $x, y$ plane for any $n \ge 2$ and any $a \ge 1$

**Solution 1.1.** We first convert from polar to Cartesian coordinates. We know the transformation from polar to Cartesian is given by

$$(r, \theta) \to \varphi(x, y) = (\sqrt{x^2 + y^2}, \arctan(y/x))$$

One can guess

$$rdrd\theta = dxdy \implies drd\theta = \frac{1}{r}dxdy = \frac{1}{\sqrt{x^2 + y^2}}dxdy$$

To verify this, we compute the Jacobian

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2}\frac{1}{1+(y/x)^2} & \frac{1}{x}\frac{1}{1+(y/x)^2} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}}$$

We now compute the integral in the problem by applying this change of variables

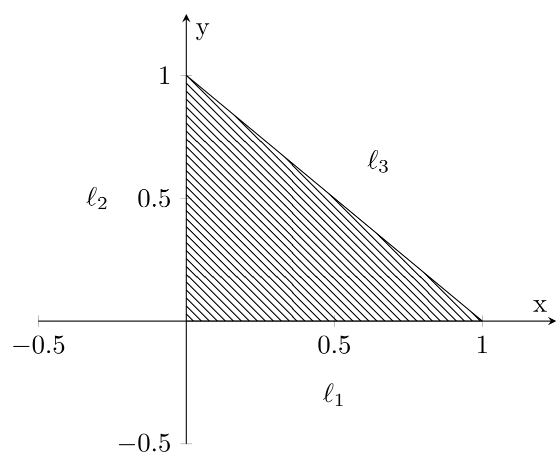
$$\begin{aligned}\iint_D r\cot^n\theta\, drd\theta &= \int_1^a\int_1^a \sqrt{x^2+y^2}(\tan(\arctan(y/x)))^{-n}\frac{1}{\sqrt{x^2+y^2}}\,dxdy \\ &= \int_1^a\int_1^a \frac{x^n}{y^n}\,dxdy \\ &= \left[\int_1^a x^n dx\right]\left[\int_1^a y^{-n}dy\right] \\ &= \left[\frac{a^{n+1}-1}{n+1}\right]\left[\frac{a^{1-n}-1}{1-n}\right].\end{aligned}$$

**Problem 1.2.** Show that

$$\int_0^1\int_0^{1-x} e^{y/(x+y)}dydx = \frac{e-1}{2}$$

Using the change of variables $u = x+y$, $y = uv$

**Solution 1.2.** We first sketch our initial region of integration. It is a triangle in $\mathbb{R}^2$. We denote the sides of the triangle $\ell_1, \ell_2, \ell_3$ Our change of variables can be defined as the non-linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$




given by

$$\varphi(x,y) = \left(x+y, \frac{y}{x+y}\right)$$

1. $\ell_1$: We first consider the image of $\ell_1 = \{(t,0) : t \in [0,1]\}$ under the map $\phi$. We just plug in $x = t$, $y = 0$ in our $\varphi$ for $t \in [0,1]$

$$\varphi(\ell_1) = \{(t,0) : t \in [0,1]\}.$$

The image is a horizontal line $v = 0$ for $u \in [0,1]$.

2. $\ell_2$: We now compute the image of $\ell_2 = \{(0, t) : t \in [0, 1]\}$ under the map $\phi$. We just plug in $x = 0$, $y = t$ in our $\varphi$ for $t \in [0, 1]$

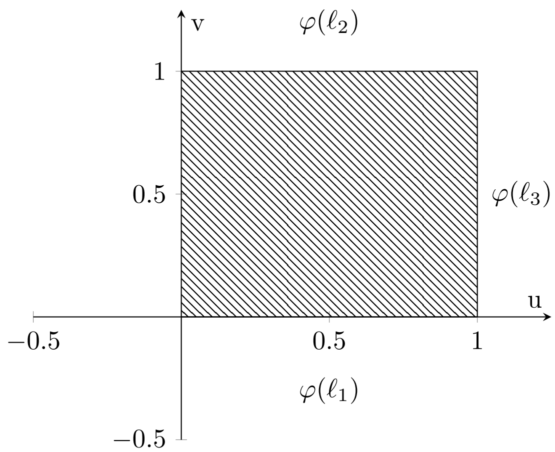
$$\varphi(\ell_2) = \{(t, 1) : t \in [0, 1]\}.$$

   The image is a horizontal line $v = 1$ for $u \in [0, 1]$.

3. $\ell_3$: We first compute the image of $\ell_3 = \{(t, 1 - t) : t \in [0, 1]\}$ under the map $\phi$. We just plug in $x = t$, $y = 1 - t$ in our $\varphi$ for $t \in [0, 1]$

$$\varphi(\ell_3) = \{(1, 1 - t) : t \in [0, 1]\}.$$

   The image is a horizontal line $u = 1$ for $v \in [0, 1]$.




The image of the shaded region under the map $\varphi$ is the region 'enclosed' by the lines $\varphi(\ell_1), \phi(\ell_2), \phi(\ell_3)$. We actually don't have a closed square because the line $\{(0, t) : t \in [0, 1]\}$ is not part of the image.

We now compute the integral using the change of variables $x = u - uv$, $y = uv$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u(1 - v) + uv = u.$$

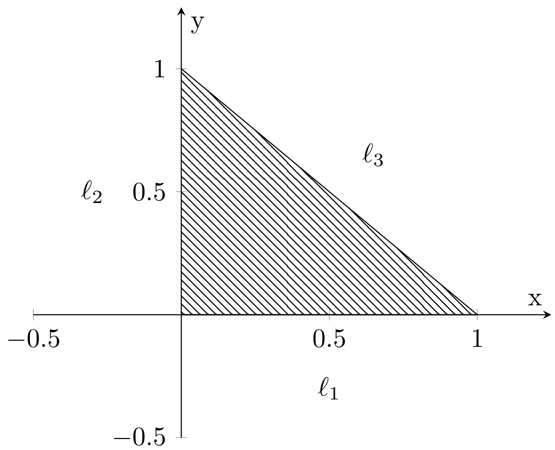
By the change of variables formula, we have

$$\begin{aligned} \int_0^1 \int_0^{1-x} e^{y/(x+y)} \, dydx &= \int_0^1 \int_0^1 e^{\frac{uv}{u - uv + uv}} |u| \, dudv \\ &= \int_0^1 \int_0^1 e^v u \, dudv \\ &= \left[ \frac{u^2}{2} \Big|_{u=0}^{u=1} \right] \left[ e^v \Big|_{v=0}^{v=1} \right] \\ &= \frac{e - 1}{2}. \end{aligned}$$

**Problem 1.3.** Let $D$ be the region bounded by $x + y = 1$, $x = 0$, $y = 0$. Show

$$\iint_D \cos\left(\frac{x - y}{x + y}\right) dxdy = \frac{\sin 1}{2}$$

Using the change of variables $u = x - y$, $v = x + y$




**Solution 1.3.** We first sketch our initial region of integration. It is a triangle in $\mathbb{R}^2$. We denote the sides of the triangle $\ell_1, \ell_2, \ell_3$ Our change of variables can be defined as the linear map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi(x, y) = (x - y, x + y)$$

(this is actually a rotation and dilation, as we will see later)

1. $\ell_1$: We first consider the image of $\ell_1 = \{(t, 0) : t \in [0, 1]\}$ under the map $\phi$. We just plug in $x = t$, $y = 0$ in our $\varphi$ for $t \in [0, 1]$

$$\varphi(\ell_1) = \{(t, t) : t \in [0, 1]\}.$$

   The image is a diagonal line from $(0, 0) \to (1, 1)$.

2. $\ell_2$: We now compute the image of $\ell_2 = \{(0, t) : t \in [0, 1]\}$ under the map $\varphi$. We just plug in $x = 0$, $y = t$ in our $\varphi$ for $t \in [0, 1]$

$$\varphi(\ell_2) = \{(-t, t) : t \in [0, 1]\}.$$

   The image is a diagonal line from $(0, 0) \to (-1, 1)$.

3. $\ell_3$: We first compute the image of $\ell_3 = \{(t, 1 - t) : t \in [0, 1]\}$ under the map $\phi$. We just plug in $x = t$, $y = 1 - t$ in our $\varphi$ for $t \in [0, 1]$

$$\varphi(\ell_3) = \{(2t - 1, 1) : t \in [0, 1]\}.$$

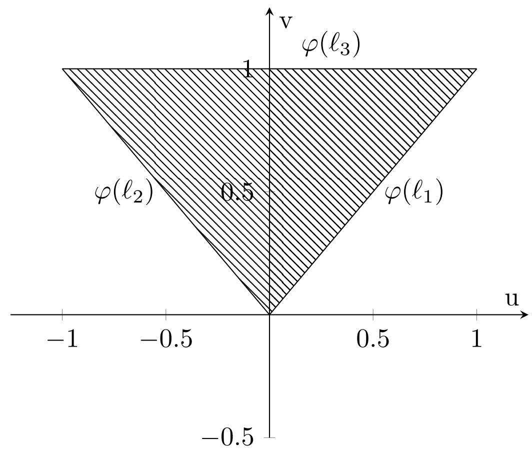
   The image is a horizontal line $(-1, 1) \to (1, 1)$.

The image of the shaded region under the map $var\phi$ is the region enclosed by the lines $\varphi(\ell_1), \varphi(\ell_2), \varphi(\ell_3)$.

We now compute the integral using the change of variables $u = x - y$, $v = x + y$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2.$$

We actually solved the change of variables the reverse direction, that is, we showed $dudv = 2dxdy$. Clearly, $dxdy = \frac{1}{2}dudv$. By the change of variables formula, we have (writing the iterated integral as




a Type 2 integral)

$$\begin{aligned}\iint_D \cos\left(\frac{x-y}{x+y}\right)\,dxdy &= \frac{1}{2}\int_0^1\int_{-v}^{v}\cos\left(\frac{u}{v}\right)\,dudv\\ &= \frac{1}{2}\int_0^1 \sin\left(\frac{u}{v}\right)v\Big|_{u=-v}^{u=v}\,dv\\ &= \frac{1}{2}\int_0^1 \sin(1)v-\sin(-1)v\,dv\\ &= \sin(1)\int_0^1 v\,dv\\ &= \frac{\sin(1)}{2}.\end{aligned}$$

# 2 Linear Algebra Review

We will review some properties of (real valued) matrices.

## 2.1 Matrix Multiplication

**Definition 2.1.** Given matrix $\boldsymbol{A} \in \mathbb{R}^{n\times p}$ and $\boldsymbol{B} \in \mathbb{R}^{p\times m}$, the product of the matrices

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} \in \mathbb{R}^{n\times m}$$

has entries given by

$$c_{ij} = (\boldsymbol{A}\boldsymbol{B})_{i,j} = a_{i1}b_{1j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj} \qquad i = 1,\dots,n, \quad j = 1,\dots,m.$$

## 2.2 Eigenvalues and Eigenvectors

**Definition 2.2.** Given a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the **eigenvalues** of $\lambda$ is a number $\lambda \in \mathbb{C}$ and the eigenvectors is a vector $\mathbf{v} \in \mathbb{C}^n$ such that

$$\boldsymbol{A}\mathbf{v} = \lambda\mathbf{v}.$$

## 2.3 Symmetric Matrices

**Definition 2.3.** A real valued matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is **symmetric** if

$$\boldsymbol{A} = \boldsymbol{A}^\top.$$

Or equivalently, $A_{ij} = A_{ji}$ for all $1 \le i, j \le n$.

**Definition 2.4.** A real valued matrix $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ is **diagonal** if all of its off-diagonal entries are zero, that is $A_{ij} = 0$ for all $i \neq j$.

**Definition 2.5.** A real valued matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ is **orthogonal** if its columns and rows are orthonomal vectors

$$\boldsymbol{Q}^\top\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^\top = \boldsymbol{I}.$$

The following result states that real symmetric matrices are determined by its eigenvalues and eigenvectors.

**Proposition 2.6**

*If $\boldsymbol{A}$ is a real symmetric matrix, then $\boldsymbol{A}$ can be decomposed into*

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^\intercal$$

*where $\boldsymbol{Q}$ is a orthgonal matrix whose columns are the orthonomal eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{D}$ is diagonal whose entries are the eigenvalues of $\boldsymbol{A}$.*

## 2.4 Positive Semidefinite Matrices

**Definition 2.7.** Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric real valued matrix. We say that $\boldsymbol{A}$ is **positive semidefinite** if

$$\mathbf{x}^\top\boldsymbol{A}\mathbf{x} \ge 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

**Proposition 2.8**

*The following are equivalent*

1. *The symmetric matrix $\boldsymbol{A}$ is real valued and positive semidefinite*
2. *The matrix $\boldsymbol{A}$ has non-negative eigenvalues*
3. *The matrix $\boldsymbol{A}$ is the covariance matrix of some multivariate distribution in $\mathbb{R}^n$*
4. *There exists a real matrix $\boldsymbol{B}$ such that $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{B}^\intercal$*

We also have the following useful properties

**Proposition 2.9 (*Properties of Positive Semidefinite Matrices*)**

1. *If $r \geq 0$ and $\boldsymbol{A}$ is positive semidefinite matrices, then*
$$r\boldsymbol{A}$$
*is also positive semidefinite.*
2. *If $\boldsymbol{A}$* and *$\boldsymbol{B}$ are two positive semidefinite matrices, then*
$$\boldsymbol{A} + \boldsymbol{B}$$
*is also positive semidefinite.*